FIITJEE LIMITED

# PRE-RMO-2018

**Time : 3 hours** **August 19, 2018**

**1.** A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number of pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n?

**Sol.** **17**

Let number of pages in volume 1 be 'x'

$\Rightarrow 1 + (x + 1) + (2x + 51) = 1709$

$\Rightarrow x = 552$

$\Rightarrow n = 2057 = 11 \times 11 \times 17$

$\Rightarrow$ Largest prime factor = 17

**2.** In a quadrilateral ABCD, it is given that AB = AD = 13, BC = CD = 20, BD = 24. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

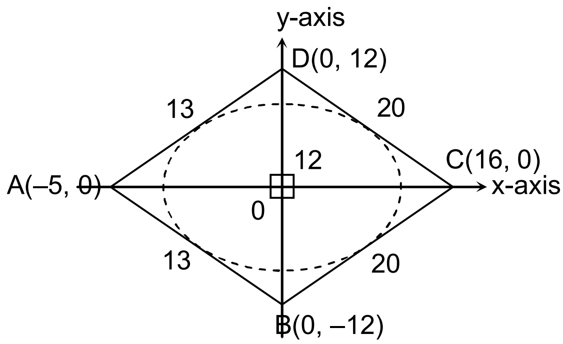
**Sol.** **8**

Sides of quadrilateral are $\begin{cases} 3x + 4y = 48 \\ 3x - 4y = 48 \\ 12x + 5y = -60 \\ 12x - 5y = -60 \end{cases}$

$$\frac{|3h + 4k - 48|}{5} = r = \frac{|3h - 4k - 48|}{5}$$

$$\frac{|12h - 5k + 60|}{13} = r = \frac{|12h + 5k + 60|}{13}$$

$\Rightarrow 33r = 60 + 48 \times 4$




**3.** Consider all 6-digit numbers of the form abccba where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.

**Sol.** **70**

If abccba is divisible by 7,

then a + 3b + 2c + 6c + 4b + 5a is divisible by '7'

$\Rightarrow$ 7b + 6a + 8c is divisible by '7'

$\Rightarrow$ c – a is divisible by '7' and 'b' is odd

$\Rightarrow (5 + 9) \times 5 = 70$ such numbers are possible

**4.** The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is, 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.

**Sol.** **12**

$(b^2 + 6b + 6)(5b + 6) = (8b^3 + 5b^2 + 9b)$

$\Rightarrow 3b^3 - 31b^2 - 57b - 36 = 0$

$\Rightarrow (b - 12)(3b^2 + 5b + 3) = 0$

$\Rightarrow b = 12$ is the only possibility

5. Let ABCD be a trapezium in which AB || CD and AD $\perp$ AB. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49, find PQ.

**Sol.** **84**

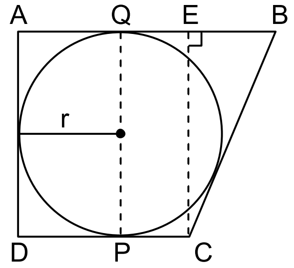
Let radius of circle b 'r'

$\Rightarrow CE^2 + BE^2 = CB^2$

$\Rightarrow (49 - 36)^2 + (2r)^2 = (49 + 36)^2$

$\Rightarrow r = 42$

$\Rightarrow PQ = 2r = 84$ units




6. Integers a, b, c satisfy $a + b - c = 1$ and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

**Sol.** **18**

$a + b - c = 1$

$\Rightarrow a^2 + b^2 - c^2 = 1 + 2c - 2ab = -1$

$\Rightarrow c = ab - 1$

$\Rightarrow a + b = ab$

$\Rightarrow (a - 1)(b - 1) = 1$

As, a, b, c are integers

$\Rightarrow a = b = 2, c = 3$

$\Rightarrow a^2 + b^2 + c^2 = 17$

and $a = b = 0, c = -1$

Sum of possible value of $a^2 + b^2 + c^2$ is 18

7. A point P in the interior of a regular hexagon is at distances 8, 8, 16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r?

**Sol.** **14**

$AB = 16\cos\alpha = BC$

$\angle CBP = 120^\circ - \alpha$

$$\cos(120^\circ - \alpha) = \frac{8^2 + (16\cos\alpha)^2 - (16)^2}{2\cdot 8\cdot 16\cos\alpha}$$

$$\Rightarrow -\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha = \frac{1 + 4\cos^2\alpha - 4}{4\cos\alpha}$$

$$\Rightarrow -2\cos^2\alpha + 2\sqrt{3}\sin\alpha\cos\alpha = 1 - 4\sin^2\alpha$$

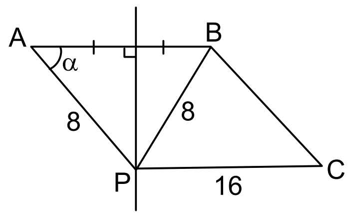
$$\Rightarrow -1 - \cos 2\alpha + \sqrt{3}\sin 2\alpha = 2\cos 2\alpha - 1$$

$$\Rightarrow \sqrt{3}\sin 2\alpha = 2\cos 2\alpha$$

$$\Rightarrow \tan 2\alpha = \sqrt{3}$$

$\Rightarrow \alpha = 30^\circ$

$r = AB = 16\cos 30^\circ \approx 13.85$

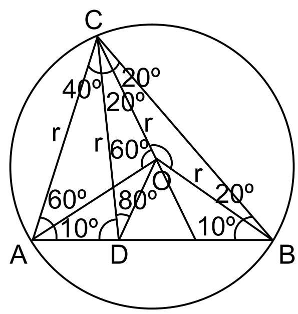



8. Let AB be a chord of a circle with centre O. Let C be a point on the circle such that $\angle ABC = 30^\circ$ and O lies inside the triangle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20^\circ$. Find the measure of $\angle CDO$ in degrees.

**Sol.** **80**

Shown in the figure

$\angle CDO = 80^\circ$



**9.** Suppose a, b are integers and a + b is a root of $x^2 + ax + b = 0$. What is the maximum possible value of $b^2$?

**Sol.** **81**

If a + b is a root it satisfies the equation

$\Rightarrow 2a^2 + 3ba + (b^2 + b) = 0$

Now, since 'a' is an integer, discriminant is a perfect square

So, $(b - 4 + p)(b - 4 - p) = 16$

$b - 4 + p = \pm 8$, $b - 4 - p = \pm 2$, $b - 4 + p = b - 4 - p = \pm 4$

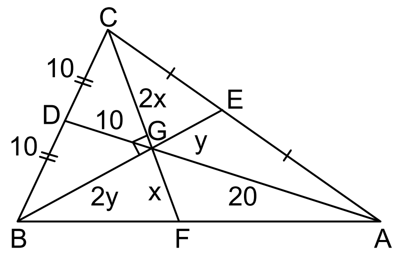
$\Rightarrow b - 4 = 5, -5, 4, -4$

$\therefore b = 9, -1, 8, 0$

$(b^2)$ maximum = 81

**10.** In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

**Sol.** **24**




$CD = BD = GD$ ($\because$ right triangle)

$AB^2 = (2BF)^2 = 4(x^2 + 4y^2)$

$AC^2 = (2CE)^2 = 4(y^2 + 4x^2)$

$BC^2 = 20^2 = 4(x^2 + y^2)$

$AB^2 + BC^2 + AC^2 = 24$

$$x^2 + 4y^2 = \frac{b^2}{4}$$

$$5\left(x^2 + y^2\right) = \frac{b^2 + c^2}{4}$$

$5(4x^2 + 4y^2) = b^2 + c^2$

Using Appoloneous theorem

$$b^2 + c^2 = 2(20)^2 + \frac{a^2}{2}$$

So, $\frac{6a^2}{100} = 24$

**11.** There are several tea cups in the kitchen, some with handles and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?

**Sol.** **29**

$n_1$ = cups with handles

$n_2$ = cups without handles

${}^{n_1}C_2 \times {}^{n_1}C_3 = 1200 \Rightarrow$ Only prime factors of ${}^{n_1}C_3$ and ${}^{n_2}C_2$ are 2, 3, 5 and ${}^{n_1}C_3 \mid 1200$

$\therefore\ {}^{n_1}C_3 \le 1200 \Rightarrow n_1 < 21$

$\Rightarrow n_1 \in \{3, 4, 5, 6, 9, 10, 16\}$

${}^{10}C_3 = 120 \Rightarrow {}^{n_2}C_2 = 10 \Rightarrow n_2 = 5$

${}^{6}C_3 = 20 \Rightarrow {}^{n_2}C_2 = 60$

$\Rightarrow$ no solution is possible

${}^{5}C_3 = 10 \Rightarrow {}^{n_2}C_2 = 120 \Rightarrow n_2 = 16$

${}^{4}C_3 = 4 \Rightarrow {}^{n_2}C_2 = 300 \Rightarrow n_2 = 25$

${}^{3}C_3 = 1 \Rightarrow {}^{n_2}C_2 = 1200$

$\Rightarrow n_2$ = no solution is possible

25 + 4 = 29

**12.** Determine the number of 8-tuples $(\in_1, \in_2, \ldots, \in_8)$ such that $\in_1, \in_2, \ldots \in_8 \in \{1, -1\}$ and

$$\in_1 + 2\in_2 + 3\in_3 + \ldots + 8\in_8$$

is a multiple of 3.

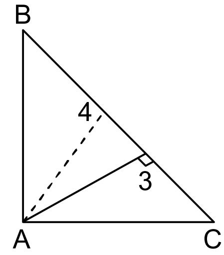
**Sol.** **88**

Expression $\equiv (\in_1 + \in_4 + \in_7) - (\in_2 + \in_5 + \in_8) \pmod 3$

Number of ways = 2 . 2 . [1 + 1 + 3 . 3 . 2 + 1 + 1] = 88

**13.** In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A.

**Sol.** **24**




$4 = \dfrac{\sqrt{2}\,bc}{b+c}$ and $3 = \dfrac{bc}{a}$

$(b + c)^2 = \dfrac{1}{8}b^2c^2 \Rightarrow b^2 + c^2 + 2bc = \dfrac{1}{8}b^2c^2$

$\Rightarrow a^2 + 6a = \dfrac{9a^2}{8} \Rightarrow \dfrac{a^2}{8} = 6a$

$\Rightarrow a = 48 \therefore m_a = \dfrac{a}{2} = 24$

**14.** If x = cos1° cos2° cos3° …. cos89° and y = cos2° cos6° cos10° …. cos86°, then what is the integer nearest to $\dfrac{2}{7}\log_2(y/x)$?

**Sol.** **19**

x = cos 1º cos 2º cos 3º ..... cos 89º

= (cos 1º sin 1º)(cos 2º sin 2º)(cos 3º sin 3º) ..... (cos 44º sin 44º)cos 45º

$= \dfrac{1}{2^{44}}\sin 2^\circ \sin 4^\circ \sin 6^\circ .....\sin 88^\circ \dfrac{1}{\sqrt{2}}$

$= \dfrac{1}{2^{44}\sqrt{2}}(\sin 2^\circ \cos 2^\circ)(\sin 4^\circ \cos 4^\circ).....(\sin 44^\circ \cos 44^\circ)$

$= \dfrac{1}{2^{44}\sqrt{2}\cdot 2^{22}}\sin 4^\circ \sin 8^\circ .....\sin 88^\circ$

$= \dfrac{1}{2^{66}\cdot\sqrt{2}}\cos 2^\circ \cos 6^\circ .....\cos 86^\circ$

$\therefore \dfrac{y}{x} = 2^{66}\cdot\sqrt{2} = 2^{66.5}$

$\therefore \dfrac{2}{7}\log_2\dfrac{y}{x} = \dfrac{2}{7}\cdot\dfrac{133}{2} = 19$

**15.** Let a and b be natural numbers such that 2a − b, a − 2b and a + b are all distinct squares. What is the smallest possible value of b?

**Sol.** **21**

$2a - b = k_1^2$ …(1)

$a - 2b = k_2^2$ …(2)

$a + b = k_3^2$ …(3)

Clearly $k_2^2 + k_3^2 = k_1^2$ ($k_2 < k_3$)

For minimum 'b' difference of $k_3^2$ and $k_2^2$ is also minimum and must be multiple of 3

So, $k_3^2 - k_2^2 = 3b = 144 - 81 = 63$

So, minimum b is 21

**16.** What is the value of $\sum\limits_{\substack{1\le i<j\le 10\\ i+j=\text{odd}}}(i+j) - \sum\limits_{\substack{1\le i<j\le 10\\ i+j=\text{even}}}(i+j)$?

**Sol.** **55**

$3 \times 1 + 5 \times 2 + 7 \times 3 + 9 \times 4 + 11 \times 5 + 13 \times 4 + 15 \times 3 + 17 \times 2 + 19 \times 1$

$-1 \times 4 - 2 \times 6 - 3 \times 8 - 4 \times 10 - 4 \times 12 - 3 \times 14 - 2 \times 16 - 1 \times 18$

= 3 + 10 + 21 _ 36 + 55 + 52 + 45 + 34 + 19

–4 – 12 – 24 – 40 – 48 – 42 – 32 – 18 = 55

**17.** Triangles ABC and DEF are such that $\angle A = \angle D$, AB = DE = 17, BC = EF = 10 and AC − DF = 12. What is AC + DF?

**Sol.** **30**

$b^2 - 2bc\cos A + c^2 - a^2 = 0 \Rightarrow b^2 - 2b(17)\cos A + (17 - 10)(17 + 10) = 0$

$|b_1 - b_2| = \sqrt{17^2 \times 4\cos^2 A - 4 \times 7 \times 27} = 12 \Rightarrow \cos^2 A = \frac{36 + 27 \times 7}{17^2} \Rightarrow \cos A = \frac{15}{17}$

$b_1 + b_2 = 2 .17 \cos A = 30$

**18.** If a, b, c $\geq$ 4 are integers, not all equal and 4abc = (a + 3) (b + 3) (c + 3), then what is the value of a + b + c?

**Sol.** **16**

Quickly looking at minimum cases gives a = 4, b = 5, c = 7

**19.** Let N = 6 + 66 + 666 + ….. + 666 ….. 66, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N?

**Sol.** **33**

$N = \frac{6}{9}((10 - 1) + (10^2 - 1) + \ldots\ldots + (10^{100} - 1))$

$= \frac{6}{9}\left[10^2\left(\frac{10^{99} - 1}{10 - 1}\right) - 90\right] = \frac{200}{27}\left(10^{99} - 1\right) - 60$

$= \frac{1}{3} \times 222 \ldots\ldots 22200 - 60 = 740740 \ldots\ldots - 60$

7 appears 33 times

**20.** Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.

**Sol.** **17**

Number of digits cannot be more than 2 as considering function

$f(n) = 10^{2(n-1)} - 15 \times 10^{n-1} - 27 - 9^n$ is always positive if $n \geq 3$

For two digit number product cannot be more than 81

So we can easily see that required number is only 17

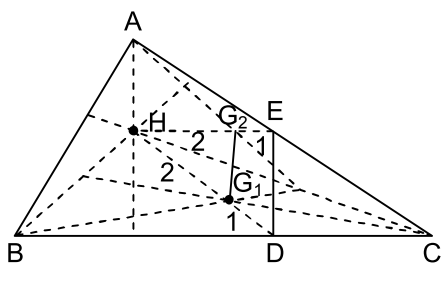
**21.** Let ABC be an acute-angled triangle and let H be its orthocentre. Let $G_1$, $G_2$ and $G_3$ be the centroids of the triangles HBC, HCA and HAB respectively. If the area of triangle $G_1G_2G_3$ is 7 units, what is the area of triangle ABC?

**Sol.** **63**

$\because \Delta G_1G_2G_3 \sim \Delta ABC$

$\Rightarrow \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta G_1G_2G_3} = \frac{(AB)^2}{(G_1G_2)^2} = \left(\frac{3}{1}\right)^2$

$\Rightarrow$ Area of $\Delta ABC = 9 \times 7 = 63$




**22.** A positive integer k is said to be good if there exists a partition of {1, 2, 3, …… 20} in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there?

**Sol.** **6**

Number of sets is less than 11 and needs to divide 210

Partition into 10 sets is trivial 2 and 5 sets.

We can now combine them to produce for partition into 5 sets use {(20, 25), (19, 16) .....} 3 sets

We can now combine them to produce for partition into 7 sets use {(20, 10), (19, 11) .....} 3 sets

Hence, the number of good numbers is 6

**23.** What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n(a + b + c)$$

holds for all positive real numbers a, b, c.

**Sol.** **14**

$$\left(\frac{a}{\sqrt{x}}\cdot\sqrt{x}+\frac{b}{\sqrt{y}}\cdot\sqrt{y}+\frac{c}{\sqrt{z}}\cdot\sqrt{z}\right)^2\left(\sum\frac{a^2}{x}\right)\left(\sum x\right)$$

$$\sum\frac{a^2}{x}\geq\frac{(a+b+c)^2}{\sum x}\geq\frac{29\times31}{60}(a+b+c)$$

(Where $x=\frac{b}{29}+\frac{c}{31}$, $y=\frac{c}{29}+\frac{a}{31}$, $z=\frac{a}{29}+\frac{b}{31}$)

n = 14

**24.** If N is the number of triangles of different shapes (i.e., not similar) whose angles are all integers (in degrees), what is N/100?

**Sol.** **27**

Let triangle's angle are A, B, C so (A, B, C)

$A^\circ + B^\circ + C^\circ = 180^\circ$ total solution of equation ${}^{179}C_2$

(I) If A = B = C = 60 – one solution

(II) If $A \neq B = C$ or $A = B \neq C$ or $A = C \neq B$ type solution

2A + B = 180 total 89 solution

So total solutions in this case = $3 \times 89 - 3 = 3 \times 88$

(III) If $A \neq B \neq C$, total solution in this case are

$$\Rightarrow \frac{{}^{179}C_2 - 3\times 88 - 1}{6} = 2611$$

So total solutions are 1 + 88 + 2611 = 2700

N = 27

**25.** Let T be the smallest positive integer which, when divided by 11, 13, 15 leaves remainders in the sets {7, 8, 9}, {1, 2, 3}, {4, 5, 6} respectively. What is the sum of the squares of the digits of T?

**Sol.** **81**

For $x \equiv 8 \pmod{11}$, $x \equiv 2 \pmod{13}$ we get 184 which satisfies the given conditions. One can check that no smaller number works

$\therefore$ sum = $1^2 + 8^2 + 4^2 = 81$

**26.** What is the number of ways in which one can choose 60 unit squares from a $11 \times 11$ chessboard such that no two chosen squares have a side in common?

**Sol.** **62**

$(6 \times 6 + 5 \times 5) + 1 = 62$

**27.** What is the number of ways in which one can colour the squares of a $4 \times 4$ chessboard with colours red and blue such that each row as well as each column has exactly two red squares and two blue squares?

**Sol.** **90**

Fix first row in ${}^4C_2$ ways

Case 1. Next row is exact opposite of row 1 and then row 3 is fixed arbitrarily $\Rightarrow 1.{}^4C_2.1$ ways

Case 2. Next row has one red in same column as first and then row 3 has one red in the empty column $\Rightarrow ({}^2C_1.{}^2C_1).1.2$ ways

Case 3. Next row is exact copy of row 1 and then row 3 & 4 are fixed $\Rightarrow$ 1 way

**28.** Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N?

**Sol.** **24**

Total methods are either 1, 3, 4 or 1, 2 5

so ways are = $\frac{8!}{1!\,2!\,5!}\times 3! + \frac{8!}{1!\,3!\,4!}\times 3! = 2688$

sum of digits = 24

**29.** Let D be an interior point of the side BC of a triangle ABC. Let $I_1$ and $I_2$ be the incentres of triangles ABD and ACD respectively. Let $AI_1$ and $AI_2$ meet BC in E and F respectively. If $\angle BI_1E = 60°$, what is the measure of $\angle CI_2F$ in degrees?

**Sol.** **30**

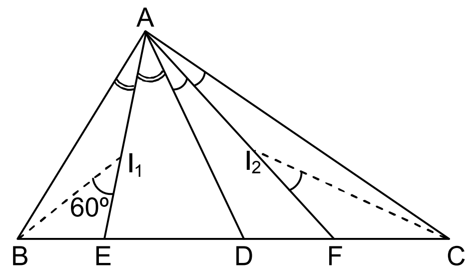
$\angle BAD = \alpha_1$ and $\angle DAC = \alpha_2$

$\angle FI_2C = \frac{\gamma}{2} + \frac{\alpha_2}{2} = x$

$\frac{\beta}{2} + \frac{\alpha_1}{2} = 60^o$

$\Rightarrow \frac{\alpha_1 + \alpha_2}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^o = 60^o + x$

$\Rightarrow x = 30^o$




**30.** Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ be a polynomial in which $a_i$ is a non-negative integer for each $i \in \{0, 1, 2, 3, \ldots., n\}$. If P(1) = 4 and P(5) = 136, what is the value of P(3)?

**30.** **34**

$P(x) = x^3 + 3x + 1$

$P(3) = 34$

******* End *******