CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
22nd May 2022

Important information and instructions:

(1) Questions in Part A (Questions 1 – 10) will be used for screening. There will be a cut-off for Part A, which will not be more than 20 marks (out of 40).

(2) Each question in Part A has one or more correct answers. Enter your answers to these questions into the computer as instructed. Every question is worth 4 marks. A solution will receive credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(3) Your solutions to the questions in Part B (Questions 11– 20*) will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.)

(4) Answer 6 questions from Part B, on the pages assigned to them, with sufficient justification. Each question is worth 10 marks. Clearly indicate which six questions you would like us to mark in the six boxes on the front sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

(5) The scores in both the sections will be taken into account while making the final decision. You are advised to spend about 90 minutes on Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least 15 marks from among the starred questions 17^*–20^*.

(6) Time: 3 hours.

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, and of complex numbers. For a field F and a positive integer n, $\mathrm{GL}(n, F)$ stands for the set of invertible $n \times n$ matrices over F. I_n denotes the $n \times n$ identity matrix; the field will be clear from context. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}$ are taken with the euclidean topology.

Part A

(1) By a *simple* group, we mean a group G in which the only normal subgroups are $\{1_G\}$ and G. Pick the correct statement(s) from below.
   (A) No group of order 625 is simple.
   (B) $\mathrm{GL}(2, \mathbb{R})$ is simple.
   (C) Let G be a simple group of order 60. Then G has exactly six subgroups of order 5.
   (D) Let G be a group of order 60. Then G has exactly seven subgroups of order 3.

(2) Let $f : \mathbb{R} \longrightarrow (0, \infty)$ be an infinitely differentiable function with $\int_{-\infty}^{\infty} f(t)dt = 1$. Pick the correct statement(s) from below.
   (A) $f(t)$ is bounded.
   (B) $\lim_{|t| \longrightarrow \infty} f'(t) = 0$.
   (C) There exists $t_0 \in \mathbb{R}$ such that $f(t_0) \geq f(t)$ for all $t \in \mathbb{R}$.
   (D) $f''(a) = 0$ for some $a \in \mathbb{R}$.

(3) Let $\mathcal{P}_n = \{f(x) \in \mathbb{R}[x] \mid \deg f(x) \leq n\}$, considered as an $(n+1)$-dimensional real vector space. Let T be the linear operator $f \mapsto f + \frac{\mathrm{d}f}{\mathrm{d}x}$ on $\mathcal{P}_n$. Pick the correct statement(s) from below.
   (A) T is invertible.
   (B) T is diagonalizable.
   (C) T is nilpotent.
   (D) $(T - I)^2 = (T - I)$ where I is the identity map.

(4) Pick the correct statement(s) from below.
   (A) There exists a finite commutative ring R of cardinality 100 such that $r^2 = r$ for all $r \in R$.
   (B) There is a field K such that the additive group $(K, +)$ is isomorphic to the multiplicative group $(K^\times, \cdot)$.

(C) An irreducible polynomial in $\mathbb{Q}[x]$ is irreducible in $\mathbb{Z}[x]$.
(D) A monic polynomial of degree n over a commutative ring R has at most n roots in R.

(5) Pick the correct statement(s) from below.
(A) if f is continuous and bounded on $(0,1)$, then f is uniformly continuous on $(0,1)$.
(B) If f is uniformly continuous on $(0,1)$, then f is bounded on $(0,1)$.
(C) If f is continuous on $(0,1)$ and $\lim_{x\longrightarrow 0^+} f(x)$ and $\lim_{x\longrightarrow 1^-} f(x)$ exists, then f is uniformly continuous on $(0,1)$.
(D) Product of a continuous and a uniformly continuous function on $[0,1]$ is uniformly continuous.

(6) Let X be the metric space of real-valued continuous functions on the interval $[0,1]$ with the "supremum distance":

$$d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\} \text{ for all } f,g \in X.$$

Let $Y = \{f \in X : f([0,1]) \subset [0,1]\}$ and $Z = \{f \in X : f([0,1]) \subset [0,\frac{1}{2}) \cup (\frac{1}{2},1]\}$. Pick the correct statement(s) from below.
(A) Y is compact.
(B) X and Y are connected.
(C) Z is not compact.
(D) Z is path-connected.

(7) Let $X := \{(x,y,z) \in \mathbb{R}^3 \mid z \leq 0, \text{ or } x,y \in \mathbb{Q}\}$ with subspace topology. Pick the correct statement(s) from below.
(A) X is not locally connected but path connected.
(B) There exists a surjective continuous function $X \longrightarrow \mathbb{Q}_{\geq 0}$ (the set of non-negative rational numbers, with the subspace topology of $\mathbb{R}$).
(C) Let S be the set of all points $p \in X$ having a compact neighbourhood (i.e. there exists a compact $K \subset X$ containing p in its interior). Then S is open.
(D) The closed and bounded subsets of X are compact.

(8) Consider the complex polynomial $P(x) = x^6 + ix^4 + 1$. (Here i denotes a square-root of -1.) Pick the correct statement(s) from below.
(A) P has at least one real zero.
(B) P has no real zeros.
(C) P has at least three zeros of the form $\alpha + i\beta$ with $\beta < 0$.
(D) P has exactly three zeros $\alpha + i\beta$ with $\beta > 0$.

(9) Let v a (fixed) unit vector in $\mathbb{R}^3$. (We think of elements of $\mathbb{R}^n$ as column vectors.) Let $M = I_3 - 2vv^t$. Pick the correct statement(s) from below.
(A) 0 is an eigenvalue of M.
(B) $M^2 = I_3$.
(C) 1 is an eigenvalue of M.
(D) The eigenspace for the eigenvalue -1 is 2-dimensional.

(10) Let $f(z) = \sum_{n\geq 0} a_n z^n$ be an analytic function on the open unit disc D around 0 with $a_1 \neq 0$. Suppose that $\sum_{n\geq 2} |na_n| < |a_1|$. Then which of the following are true?
(A) There are only finitely many such f.
(B) $|f'(z)| > 0$ for all $z \in D$.
(C) If $z, w \in D$ are such that $z \neq w$ and $f(z) = f(w)$, then $a_1 = -\sum_{n\geq 2} a_n(z^{n-1} + z^{n-2}w + \cdots + w^{n-1})$.
(D) f is one-one on D.

Part B

(11) Let $A \in \mathrm{GL}(3, \mathbb{Q})$ with $A^t A = I_3$. Assume that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for some $\lambda \in \mathbb{C}$.

(A) Determine the possible values of λ.

(B) Determine $x + y + z$ where x, y, z is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

(12) Consider the function $S(a)$ defined by the limit below:

$$S(a) := \lim_{n \longrightarrow \infty} \frac{1^a + 2^a + 3^a + \cdots + n^a}{(n+1)^{a-1}[(na+1) + (na+2) + \cdots + (na+n)]}.$$

Find the sum of all values a such that $S(a) = \frac{1}{60}$.

(13) Let U and V be non-empty open connected subsets of $\mathbb{C}$ and $f : U \longrightarrow V$ an analytic function. Suppose that for all compact subsets K of V, $f^{-1}(K)$ is compact. Show that $f(U) = V$.

(14) Let G be a finite group that has a non-trivial subgroup N (i.e. $\{1_G\} \neq N \neq G$) that is contained in every non-trivial subgroup of G. Show that

(A) G is a p-group for some prime number p;

(B) N is a normal subgroup of G.

(15) Let f be an entire function such that f maps the open unit ball D around 0 to itself. Suppose further that $f(0) = 0$ and $f(1) = 1$. Show that $f'(1) \in \mathbb{R}$ and that $|f'(1)| \geq 1$.

(16) Let F be a field such that it has a finite non-Galois extension field. Let V be a finite-dimensional vector-space over F. Let $V_1, \ldots, V_r$ be proper subspaces of V. Prove or disprove the following assertion: $V \neq \bigcup_{i=1}^{r} V_i$.

(17*) For a ring homomorphism $R \longrightarrow S$ (of commutative rings) and an ideal I of R, the *fibre* over I is the ring S/IS, i.e., the quotient of S by the S-ideal generated by the image of I in S. Let $S = \mathbb{C}[X,Y]/(XY-1)$ and $R = \mathbb{C}[x+\alpha y]$ where $\alpha \in \mathbb{C}$ and x, y are the images of X, Y in S. Consider the ring homomorphism $R \subseteq S$. Let $I = (x + \alpha y - \beta)R$, where $\beta \in \mathbb{C}$. For each non-negative integer n, determine the set of (α, β) such that the fibre over I has exactly n maximal ideals.

(18*) Let Q be the space of infinite sequences

$$\mathbf{x} := (x_1, x_2, \ldots, x_n, \ldots)$$

of real numbers $x_n \in [0,1]$, with the product topology coming from the identification $Q = [0,1]^{\mathbb{N}}$. ($[0,1]$ has the euclidean topology.) Let $S : Q \longrightarrow \mathbb{R}$ be the map

$$S(\mathbf{x}) := \sum_n \frac{x_n}{n^2}.$$

(A) Let $Q_2 := \{(y_1, y_2, \ldots, y_n, \ldots) \mid 0 \le y_n \le \frac{1}{n}\}$. Show that Q_2 is compact.

(B) Let $D : Q_2 \longrightarrow Q$ be the map

$$(y_1, y_2, \ldots, y_n, \ldots) \mapsto (y_1, 2y_2, \ldots, ny_n, \ldots).$$

Show that D is a homeomorphism. (Hint: first show that Q is Hausdorff.)

(C) Show that $S \circ D : Q_2 \longrightarrow \mathbb{R}$ is continuous. (Hint: Show that there is a suitable inner-product space $(L, \langle -, - \rangle)$ and a vector $\mathbf{a} \in L$ such that $(S \circ D)(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a} \rangle$ for each $\mathbf{x} \in Q_2$.)

(D) Show that S is continuous.

(19*) Let E be a finite extension of the field $\mathbb{Q}$. We say that a homomorphism of fields $\phi : E \longrightarrow \mathbb{C}$ is *real* if $\phi(E) \subset \mathbb{R}$. Prove or disprove each of the following assertions:

(A) For each prime number p, the field $\mathbb{Q}(p^{1/12})$ has exactly one real embedding in $\mathbb{C}$. ($p^{1/12}$ is the unique real number $r > 0$ such that $r^{12} = p$.)

(B) If $[E : \mathbb{Q}] = 11$, there exists a real embedding of E.

(C) If E is a Galois extension of $\mathbb{Q}$ and $[E : \mathbb{Q}] = 11$, then every embedding $E \longrightarrow \mathbb{C}$ is a real embedding.

(20*) A continuous map $f : A \longrightarrow B$ between two metric space $(A, d_A), (B, d_B)$ is said to be a *bilipschitz map* if there exists a real number $\lambda > 0$ such that $(1/\lambda)d_A(a_0, a_1) \le d_B(f(a_0), f(a_1)) \le \lambda d_A(a_0, a_1)$ for each $a_0, a_1 \in A$.

Let $X = \mathbb{R}^2 \setminus \{0\}$ and $Y = \mathbb{S}^1 \times \mathbb{R} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$. Let d_X (respectively, d_Y) be the euclidean metric on X induced from $\mathbb{R}^2$ (respectively, on Y induced from $\mathbb{R}^3$). Let $f : X \longrightarrow Y$ be a bilipschitz map.

(A) Let $R > 0$ and $C_R \subseteq X$ the circle of radius R with centre at 0. Let $\bar{f} : X \longrightarrow \mathbb{R}$ be the composite of f and the projection from $Y = \mathbb{S}^1 \times \mathbb{R}$ to its second factor $\mathbb{R}$. Let L_R be the length of the interval $\bar{f}(C_R) \subseteq \mathbb{R}$. Let $a, b \in X$ be such that $\bar{f}(b) = \bar{f}(a) + L_R$. Show that $d_X(a,b) \ge (2R - 2\lambda)/\lambda^2$.

(B) Let C_1 and C_2 be the two arcs of C_R, joining a and b. Show that there exists $x_1 \in C_1$ and $x_2 \in C_2$ such that $\bar{f}(x_1) = \bar{f}(x_2) = \frac{f(a)+f(b)}{2}$. Show that $d_Y(f(x_1), f(x_2)) \le 2$.

(C) Show that for all sufficiently large R, $d_Y(f(x_1), f(x_2)) > 2$. (Hint: Let $a_i \in C_i$ be such that $d_X(a, a_i) = R/\lambda^2$; show that $d_X(x_1, x_2) \ge d_X(a_1, a_2)$.)

(D) What is your conclusion about f?