CHENNAI MATHEMATICAL INSTITUTE

Undergraduate Programme in Mathematics and Computer Science/Physics
Common Entrance Examination
1^{st} August 2021

Enter your *Admit Card Number*: U – □□□ – □□□□

IMPORTANT INSTRUCTIONS

- **Points: 40 for part A and 60 for part B.** Carefully read the specific instructions given for each part in the question paper.
- **Part A will be used for screening.** Part B will be graded only if you score a certain minimum in part A. However your individual scores in both parts will be used while making the final decision.
- **Enter your answers to part A into the computer as instructed.** Each part A question has four statements, of which *at least one* is true. You have to select **exactly** the true option(s) for each question. Deciding the truth/falsity of *all four options* correctly is worth 4 points. Getting *three out of four* correct is worth 1 point. There is no negative marking.
- **This booklet is ONLY for part B answers and rough work.** For each part B problem, write your solution on the pages designated for that problem in pages numbered 2 to 13. For extra space and rough work, use the blank pages numbered 14 to 26 at the end.
- **Time allowed: 3 hours.** You are advised to leave about 2 hours for part B.

For office use only

	Points	Remarks
Part A		
Part B		
Total		

	Points	Remarks
B1		
B2		
B3		
B4		
B5		
B6		
Total		

Part A

In each question four statements are given, of which <u>*at least one*</u> is true. Select **exactly** the true option(s) for each question. Deciding the truth/falsity of *all four options* correctly is worth 4 points. Getting three out of four correct is worth 1 point. There is no negative marking. Points will be given based only on answers entered into the computer.

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1. Consider the two equations numbered [1] and [2]:

$$\log_{2021} a = 2022 - a \qquad [1]$$
$$2021^b = 2022 - b \qquad [2]$$

   (a) Equation [1] has a unique solution.
   (b) Equation [2] has a unique solution.
   (c) There exists a solution a for [1] and a solution b for [2] such that $a = b$.
   (d) There exists a solution a for [1] and a solution b for [2] such that $a+b$ is an integer.

2. A prime p is an integer ≥ 2 whose only positive integer factors are 1 and p.

   (a) For any prime p the number $p^2 - p$ is always divisible by 3.
   (b) For any prime $p > 3$ exactly one of the numbers $p - 1$ and $p + 1$ is divisible by 6.
   (c) For any prime $p > 3$ the number $p^2 - 1$ is divisible by 24.
   (d) For any prime $p > 3$ one of the three numbers $p + 1, p + 3$ and $p + 5$ is divisible by 8.

3. We want to construct a triangle ABC such that angle A is 20.21°, side AB has length 1 and side BC has length x where x is a positive real number. Let $N(x) =$ the number of pairwise noncongruent triangles with the required properties.

   (a) There exists a value of x such that $N(x) = 0$.
   (b) There exists a value of x such that $N(x) = 1$.
   (c) There exists a value of x such that $N(x) = 2$.
   (d) There exists a value of x such that $N(x) = 3$.

4. Consider polynomials of the form $f(x) = x^3 + ax^2 + bx + c$ where a, b, c are *integers*. Name the three (possibly non-real) roots of $f(x)$ to be p, q, r.

   (a) If $f(1) = 2021$, then $f(x) = (x-1)(x^2+sx+t)+2021$ where s, t must be integers.

   (b) There is such a polynomial $f(x)$ with $c = 2021$ and $p = 2$.

   (c) There is such a polynomial $f(x)$ with $r = \frac{1}{2}$.

   (d) The value of $p^2 + q^2 + r^2$ does not depend on the value of c.

5. For any *complex* number z define $P(z) =$ the cardinality of $\{z^k | k \text{ is a positive integer}\}$, i.e., the number of distinct positive integer powers of z. It may be useful to remember that π is an irrational number.

   (a) For each positive integer n there is a complex number z such that $P(z) = n$.

   (b) There is a *unique* complex number z such that $P(z) = 3$.

   (c) If $|z| \neq 1$, then $P(z)$ is infinite.

   (d) $P(e^i)$ is infinite.

6. A stationary point of a function f is a real number r such that $f'(r) = 0$. A polynomial need not have a stationary point (e.g. $x^3 + x$ has none). Consider a polynomial $p(x)$.

   (a) If $p(x)$ is of degree 2022, then $p(x)$ must have at least one stationary point.

   (b) If the number of distinct *real* roots of $p(x)$ is 2021, then $p(x)$ must have at least 2020 stationary points.

   (c) If the number of distinct *real* roots of $p(x)$ is 2021, then $p(x)$ can have at most 2020 stationary points.

   (d) If r is a stationary point of $p(x)$ AND $p''(r) = 0$, then the point $(r, p(r))$ is neither a local maximum nor a local minimum point on the graph of $p(x)$.

7. Given three *distinct positive* constants a, b, c we want to solve the simultaneous equations

$$ax + by = \sqrt{2}$$
$$bx + cy = \sqrt{3}$$

   (a) There exists a combination of values for a, b, c such that the above system has infinitely many solutions (x, y).

   (b) There exists a combination of values for a, b, c such that the above system has exactly one solution (x, y).

   (c) Suppose that for a combination of values for a, b, c, the above system has NO solution. Then $2b < a + c$.

   (d) Suppose $2b < a + c$. Then the above system has NO solution.

8. Given two *distinct nonzero* vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in 3 dimensions, define a sequence of vectors by

$$\mathbf{v}_{n+2} = \mathbf{v}_n \times \mathbf{v}_{n+1} \text{ (so } \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2, \mathbf{v}_4 = \mathbf{v}_2 \times \mathbf{v}_3 \text{ and so on).}$$

Let $S = \{\mathbf{v}_n | n = 1, 2, \ldots\}$ and $U = \{\frac{\mathbf{v}_n}{|\mathbf{v}_n|} | n = 1, 2, \ldots\}$. (**Note**: Here $\times$ denotes the cross product of vectors and $|\mathbf{v}|$ denotes the magnitude of the vector $\mathbf{v}$. The vector $\mathbf{0}$ with 0 magnitude, if it occurs in S, is counted. But in that case of course the $\mathbf{0}$ vector is not considered while listing elements of U.)

(a) There exist vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ for which the cardinality of S is 2.

(b) There exist vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ for which the cardinality of S is 3.

(c) There exist vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ for which the cardinality of S is 4.

(d) Suppose that for some $\mathbf{v_1}$ and $\mathbf{v_2}$, the set S is infinite. Then the set U is also infinite.

9.
$$f(x) = \frac{x}{x + \sin x} \qquad \text{and} \qquad g(x) = \frac{x^4 + x^6}{e^x - 1 - x^2}.$$

(a) Limit as $x \to 0$ of $f(x)$ is $\frac{1}{2}$.

(b) Limit as $x \to \infty$ of $f(x)$ does not exist.

(c) Limit as $x \to \infty$ of $g(x)$ is finite.

(d) Limit as $x \to 0$ of $g(x)$ is 720.

10. Let $f(u) = \tan^{-1}(u)$, a function whose domain in the set of all real numbers and whose range is $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $g(v) = \int_0^v f(t)dt$.

(a) $f(1) = \frac{\pi}{4}$.

(b) $f(1) + f(2) + f(3) = \pi$.

(c) g is an increasing function on the entire real line.

(d) g is an odd function, i.e., $g(-x) = -g(x)$ for all real x.

Part B

Each problem is worth 10 points. **Clearly explain your entire reasoning** unless instructed otherwise. No credit will be given without correct reasoning. Partial solutions may get partial credit. You may solve a later part of a problem by assuming a previous part, even if you could not do the earlier part.

B1. Solve the following two independent problems on pages 2–3 of the answer booklet.

(i) Let f be a function from domain S to codomain T. Let g be another function from domain T to codomain U. For each of the blanks below choose a single letter corresponding to one of the four options listed underneath. (*It is not necessary that each choice is used exactly once.*) Write your answers on page 2 as a sequence of four letters in correct order. **Do NOT explain your answers**.

If $g \circ f$ is one-to-one then f __________ and g __________.

If $g \circ f$ is onto then f __________ and g __________.

Option A: must be one-to-one and must be onto.

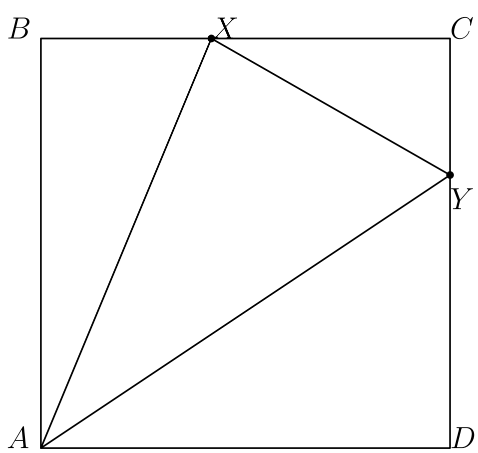
Option B: must be one-to-one but need not be onto.

Option C: need not be one-to-one but must be onto.

Option D: need not be one-to-one and need not be onto.

Recall: $g \circ f$ is the function defined by $g \circ f(a) = g(f(a))$. The function f is said to be one-to-one if, for any a_1 and any a_2 in S, $f(a_1) = f(a_2)$ implies $a_1 = a_2$. The function f is said to be onto if, for any b in T, there is an a in S such that $f(a) = b$.

(ii) In the given figure $ABCD$ is a square. Points X and Y, respectively on sides BC and CD, are such that X lies on the circle with diameter AY. What is the area of the square $ABCD$ if $AX = 4$ and $AY = 5$? (Figure is schematic and not to scale.)



B2. Solve the following two independent problems on pages 4–5 of the answer booklet.

(i) A mother and her two daughters participate in a game show. At first, the mother tosses a fair coin.

Case 1: If the result is heads, then all three win individual prizes and the game ends.

Case 2: If the result is tails, then *each* daughter separately throws a fair die and wins a prize if the result of *her* die is 5 or 6. (Note that in case 2 there are two independent throws involved and whether each daughter gets a prize or not is unaffected by the other daughter's throw.)

(a) Suppose the first daughter did not win a prize. What is the probability that the second daughter also did not win a prize?

(b) Suppose the first daughter won a prize. What is the probability that the second daughter also won a prize?

(ii) Prove or disprove each of the following statements.

(a) $2^{40} > 20!$

(b) $1 - \frac{1}{x} \leq \ln x \leq x - 1$ for all $x > 0$.

B3. You are supposed to create a 7-character long password for your mobile device.

(i) How many 7-character passwords can be formed from the 10 digits and 26 letters? (Only lowercase letters are taken throughout the problem.) Repeats are allowed, e.g., 0001a1a is a valid password.

(ii) How many of the passwords contain at least one of the 26 letters *and* at least one of the 10 digits? Write your answer in the form: (Answer to part i) − (something).

(iii) How many of the passwords contain at least one of the 5 vowels, at least one of the 21 consonants *and* at least one of the 10 digits? Extend your method for part ii to write a formula and explain your reasoning.

(iv) Now suppose that in addition to the lowercase letters and digits, you can also use 12 special characters. How many 7-character passwords are there that contain at least one of the 5 vowels, at least one of the 21 consonants, at least one of the 10 digits *and* at least one of the 12 special characters? Write *only the final formula* analogous to your answer to part iii. **Do NOT explain.**

B4. Show that there is no polynomial $p(x)$ for which $\cos(\theta) = p(\sin\theta)$ for all angles θ *in some nonempty interval.*

Hint: Note that x and $|x|$ are different functions but their values are equal on an interval (as $x = |x|$ for all $x \geq 0$). You may want to show as a first step that this cannot happen for two polynomials, i.e., if polynomials f and g satisfy $f(x) = g(x)$ for all x in some interval, then f and g must be equal as polynomials, i.e., in each degree they must have the same coefficient.

B5. Define a function f as follows: $f(0) = 0$ and, for any $x > 0$,

$$f(x) = \lim_{L\to\infty} \int_{\frac{1}{x}}^{L} \frac{1}{t^2}\cos(t)dt \ \Big(\text{or, in simpler notation, the improper integral } \int_{\frac{1}{x}}^{\infty} \frac{1}{t^2}\cos(t)dt\Big).$$

(i) Show that the definition makes sense for any $x > 0$ by justifying why the limit in the definition exists, i.e., why the improper integral converges.

(ii) Find $f'(\frac{1}{\pi})$ if it exists. Clearly indicate the basic result(s) you are using.

(iii) Using the hint or otherwise, find $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$, i.e., the right hand derivative of f at $x = 0$. We can take the limit only from the right hand side because $f(x)$ is undefined for negative values of x.

Hint: Break $f(h)$ into two terms by using a standard technique with an appropriate choice. Then separately analyze the resulting two terms in the derivative.

B6. n and k are positive integers, not necessarily distinct. You are given two stacks of cards with a number written on each card, as follows.

Stack A has n cards. On each card a number in the set $\{1, \ldots, k\}$ is written.
Stack B has k cards. On each card a number in the set $\{1, \ldots, n\}$ is written.

Numbers may repeat in either stack. From this, you play a game by constructing a sequence $t_0, t_1, t_2, \ldots$ of integers as follows. Set $t_0 = 0$. For $j > 0$, there are two cases:

If $t_j \leq 0$, draw the top card of stack A. Set $t_{j+1} = t_j +$ the number written on this card.
If $t_j > 0$, draw the top card of stack B. Set $t_{j+1} = t_j -$ the number written on this card.

In either case discard the taken card and continue. The game ends when you try to draw from an empty stack. *Example*: Let $n = 5$, $k = 3$, stack $A = 1, 3, 2, 3, 2$ and stack $B = 2, 5, 1$. You can check that the game ends with the sequence $0, 1, -1, 2, -3, -1, 2, 1$ (and with one card from stack A left unused).

(i) Prove that for every j we have $-n + 1 \leq t_j \leq k$.

(ii) Prove that there are at least two distinct indices i and j such that $t_i = t_j$.

(iii) Using the previous parts or otherwise, prove that there is a nonempty subset of cards in stack A and another subset of cards in stack B such that the sum of numbers in both the subsets is same.