MTH

GS-2018 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 10, 2017**

For the Ph.D. Programs at TIFR, Mumbai and CAM & ICTS, Bangalore and for the Int. Ph.D. Programs at TIFR, Mumbai and CAM, Bangalore.

Duration: Two hours (2 hours)

Name: ____ Ref. Code: ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill in details about name, reference code etc. on the answer sheet for. The Answer Sheet is machine-readable. Use only Black/Blue ball point pen to fill in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question: this will be treated as a wrong answer.

3. There are twenty-five (25) True/False type questions in **PART A** of the question paper. **PART B** contains 15 multiple choice questions. Questions in both Parts carry +1 for a correct answer, -1 (negative marks) for a wrong answer and 0 for not answering.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an invigilator.

6. **Use of calculators, mobile phones, laptops, tablets (or other electronic devices, including those connecting to the internet) is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilators will announce it publicly.

8. Notation and Conventions used in this test are given on page 2 of the question paper.

Test structure

The duration of this test is two hours. It has two parts, Part A and Part B. Part A has 25 'True or False' questions. Part B has 15 multiple choice questions. Each multiple choice question comes with four options, of which exactly one is correct.

Marking scheme

In both Part A and Part B, a correct answer will get 1 point, a wrong answer or an invalid answer (such as ticking multiple boxes) will get -1 point, and not attempting a particular question will get 0 points.

Notation and conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \cdots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rationals, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology and metric.
- $M_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices with the Euclidean metric, and I denotes the identity matrix.
- For any prime number p, $\mathbb{F}_p$ denotes the finite field with p elements.
- All rings are associative, with a multiplicative identity.
- All logarithms are natural logarithms.

Part A

Answer whether the following statements are True or False. Mark your answer on the machine checkable answer sheet that is provided.

1. Let A be a countable subset of $\mathbb{R}$ which is well-ordered with respect to the usual ordering on $\mathbb{R}$ (where 'well-ordered' means that every nonempty subset has a minimum element in it). Then A has an order preserving bijection with a subset of $\mathbb{N}$. F

2. $\lim_{x \to 0} \dfrac{\sin x}{\log(1 + \tan x)} = 1$. T

3. For any closed subset $A \subset \mathbb{R}$, there exists a continuous function f on $\mathbb{R}$ which vanishes exactly on A. T

4. Let f be a nonnegative continuous function on $\mathbb{R}$ such that $\int_0^\infty f(t)dt$ is finite. Then $\lim_{x \to \infty} f(x) = 0$. F

5. The function $f(x) = \cos(e^x)$ is not uniformly continuous on $\mathbb{R}$. T

6. Let A be a 3×3 real symmetric matrix such that $A^6 = I$. Then, $A^2 = I$. T

7. In the vector space $\{f \mid f : [0,1] \to \mathbb{R}\}$ of real-valued functions on the closed interval $[0,1]$, the set $S = \{\sin(x), \cos(x), \tan(x)\}$ is linearly independent. T

8. Let f be a twice differentiable function on $\mathbb{R}$ such that both f and f'' are strictly positive on $\mathbb{R}$. Then $\lim_{x \to \infty} f(x) = \infty$. F

9. Let G, H be finite groups. Then any subgroup of $G \times H$ is equal to $A \times B$ for some subgroups $A < G$ and $B < H$. F

10. Let g be a continuous function on $[0,1]$ such that $g(1) = 0$. Then the sequence of functions $f_n(x) = x^n g(x)$ converges uniformly on $[0,1]$. T

11. Let $A, B, C \in M_3(\mathbb{R})$ be such that A commutes with B, B commutes with C and B is not a scalar matrix. Then A commutes with C. F

12. If $A \in M_n(\mathbb{R})$ (with $n \geq 2$) has rank 1, then the minimal polynomial of A has degree 2. T

13. Let V be the vector space over $\mathbb{R}$ consisting of polynomials of degree less than or equal to 3. Let $T : V \to V$ be the operator sending $f(t)$ to $f(t+1)$, and $D : V \to V$ the operator sending $f(t)$ to $df(t)/dt$. Then T is a polynomial in D. T

14. Let V be the subspace of the real vector space of real valued functions on $\mathbb{R}$, spanned by $\cos t$ and $\sin t$. Let $D : V \to V$ be the linear map sending $f(t) \in V$ to $df(t)/dt$. Then D has a real eigenvalue. F

15. The set of nilpotent matrices in $M_3(\mathbb{R})$ spans $M_3(\mathbb{R})$ considered as an $\mathbb{R}$-vector space (a matrix A is said to be nilpotent if there exists $n \in \mathbb{N}$ such that $A^n = 0$). F

16. Let G be a finite group with a normal subgroup H such that G/H has order 7. Then $G \cong H \times G/H$. F

17. The multiplicative group $\mathbb{F}_7^\times$ is isomorphic to a subgroup of the multiplicative group $\mathbb{F}_{31}^\times$. T

18. Any linear transformation $A : \mathbb{R}^4 \to \mathbb{R}^4$ has a proper non-zero invariant subspace. T

19. Let $A, B \in M_n(\mathbb{R})$ be such that $A + B = AB$. Then $AB = BA$. T

20. Let $A \in M_n(\mathbb{R})$ be upper triangular with all diagonal entries 1 such that $A \neq I$. Then A is not diagonalizable. T

21. A countable group can have only countably many distinct subgroups. F

22. There exists a continuous surjection from $\mathbb{R}^3 - S^2$ to $\mathbb{R}^2 - \{(0,0)\}$ (here $S^2 \subset \mathbb{R}^3$ denotes the unit sphere defined by the equation $x^2 + y^2 + z^2 = 1$). T

23. The permutation group S_{10} has an element of order 30. T

24. Let G be a finite group and $g \in G$ an element of even order. Then we can colour the elements of G with two colours in such a way that x and gx have different colours for each $x \in G$. T

25. Let $f(x)$ and $g(x)$ be uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Then their pointwise product $f(x)g(x)$ is uniformly continuous. F

Part B

Answer the following multiple choice questions, by appropriately marking your answer on the machine checkable answer sheet that is provided.

1. The set of real numbers in the open interval $(0, 1)$ which have more than one decimal expansion is

   (a) empty.

   (b) non-empty but finite.

   (c) countably infinite.

   (d) uncountable.

2. How many zeroes does the function $f(x) = e^x - 3x^2$ have in $\mathbb{R}$?

   (a) 0

   (b) 1

   (c) 2

   (d) 3.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \textit{and} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ with } m, n \in \mathbb{Z}, n > 0, \text{ and } \gcd(m, n) = 1. \end{cases}$$

   Which of the following statements is true?

   (a) f is continuous everywhere except at 0.

   (b) f is continuous only at the irrationals.

   (c) f is continuous only at the non-zero rationals.

   (d) f is not continuous anywhere.

4. Suppose p is a degree 3 polynomial such that $p(0) = 1$, $p(1) = 2$, and $p(2) = 5$. Which of the following numbers cannot equal $p(3)$?

   (a) 0

   (b) 2

   (c) 6

   (d) 10.

5. Let A be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following two properties:

   - f has derivatives of all orders, and
   - for all $x, y \in \mathbb{R}$,

$$f(x + y) - f(y - x) = 2xf'(y).$$

   Which of the following sentences is true?

   (a) Any $f \in A$ is a polynomial of degree less than or equal to 1.
   (b) Any $f \in A$ is a polynomial of degree less than or equal to 2.
   (c) There exists $f \in A$ which is not a polynomial.
   (d) There exists $f \in A$ which is a polynomial of degree 4.

6. Denote by $\mathfrak{A}$ the set of all $n \times n$ complex matrices A ($n \geq 2$ a natural number) having the property that 4 is the only eigenvalue of A. Consider the following four statements.

   - $(A - 4I)^n = 0$,
   - $A^n = 4^n I$,
   - $(A^2 - 5A + 4I)^n = 0$,
   - $A^n = 4nI$.

   How many of the above statements are true for all $A \in \mathfrak{A}$?

   (a) 0
   (b) 1
   (c) 2
   (d) 3.

7. Let A be the set of all continuous functions $f : [0, 1] \to [0, \infty)$ satisfying the following condition:

$$\int_0^x f(t)\,dt \geq f(x), \quad \text{for all } x \in [0, 1].$$

   Then which of the following statements is true?

   (a) A has cardinality 1.
   (b) A has cardinality 2.
   (c) A is infinite.
   (d) A is empty.

8. Consider the following four sets of maps $f : \mathbb{Z} \to \mathbb{Q}$:

   (i) $\{f : \mathbb{Z} \to \mathbb{Q} \mid f \text{ is bijective and increasing}\}$,
   (ii) $\{f : \mathbb{Z} \to \mathbb{Q} \mid f \text{ is onto and increasing}\}$,
   (iii) $\{f : \mathbb{Z} \to \mathbb{Q} \mid f \text{ is bijective, and satisfies that } \forall\, n \leq 0, f(n) \geq 0\}$, and
   (iv) $\{f : \mathbb{Z} \to \mathbb{Q} \mid f \text{ is onto and decreasing}\}$.

   How many of these sets are empty?

   (a) 0
   (b) 1
   (c) 2
   (d) 3. ✓

9. What are the last 3 digits of 2^{2017}?

   (a) 072 ✓
   (b) 472
   (c) 512
   (d) 912.

10. The minimal polynomial of $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}$ is

   (a) $(x-2)(x-5)$.
   (b) $(x-2)^2(x-5)$. ✓
   (c) $(x-2)^3(x-5)$.
   (d) none of the above.

11. Consider a cube C centered at the origin in $\mathbb{R}^3$. The number of invertible linear transformations of $\mathbb{R}^3$ which map C onto itself is

   (a) 72
   (b) 48 ✓
   (c) 24
   (d) 12.

12. The number of rings of order 4, up to isomorphism, is:

    (a) 1
    (b) 2
    (c) 3
    (d) 4. ✓

13. For a sequence $\{a_n\}$ of real numbers, which of the following is a negation of the statement '$\lim_{n\to\infty} a_n = 0$'?

    (a) There exists $\varepsilon > 0$ such that the set $\{n \in \mathbb{N} \mid |a_n| > \varepsilon\}$ is infinite. ✓
    (b) For any $M > 0$, there exists $N \in \mathbb{N}$ such that $|a_n| > M$ for all $n \geq N$.
    (c) There exists a nonzero real number a such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $|a_n - a| < \varepsilon$ for all $n \geq N$.
    (d) For any $a \in \mathbb{R}$, and every $\varepsilon > 0$, there exist infinitely many n such that $|a_n - a| > \varepsilon$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then which of the following statements implies that $f(0) = 0$?

    (a) $\lim_{n\to\infty} \int_0^1 f(x)^n \, dx = 0$.
    (b) $\lim_{n\to\infty} \int_0^1 f(x/n) \, dx = 0$. ✓
    (c) $\lim_{n\to\infty} \int_0^1 f(nx) \, dx = 0$.
    (d) None of the above.

15. Consider the following maps from $\mathbb{R}^2$ to $\mathbb{R}^2$:

    (i) the map $(x, y) \mapsto (2x + 5y + 1, x + 3y)$,
    (ii) the map $(x, y) \mapsto (x + y^2, y + x^2)$, and
    (iii) the map given in polar coordinates as $(r, \theta) \mapsto (r, \theta + r^3)$ for $r \neq 0$, with the origin mapping to the origin.

    The number of maps in the above list that preserve areas is:

    (a) 0
    (b) 1
    (c) 2 ✓
    (d) 3.