Math 4.1

IV Semester M.Sc. in Mathematics Examination, June 2014
NUMBER THEORY

Time : 3 Hours   Max. Marks : 80

Note : *1) Answer* ***any five*** *questions.*
2) ***All*** *questions carry* ***equal*** *marks.*

1. a) If a and b are any two integers, not both of them are zero, then show that there exists integers x and y such that gcd (a, b) = ax + by.
   b) State and prove the fundamental theorem of arithmetic.
   c) Explain the prime number theorem. **(5+7+4)**

2. a) State and prove Chinese remainder theorem.
   b) If p is a prime, then prove that $(p-1)! \equiv -1 \pmod{p}$.
   c) Show that $18! \equiv -1 \pmod{437}$. **(8+5+3)**

3. a) If $n = p_1^{k_1} \; p_2^{k_2}, \ldots, p_r^{k_r}$ is the prime factorization of n > 1, then prove that
   i) $\tau(n) = (k_1 + 1)\,(k_2 + 2) \ldots (k_r + 1)$
   ii) $\sigma(n) = \dfrac{p_1^{k_1+1} - 1}{p_1 - 1} \; \dfrac{p_2^{k_2+1} - 1}{p_2 - 1} \; \ldots \; \dfrac{p_r^{k_r+1} - 1}{p_r - 1}$.
   b) State and prove the Mobius inversion formula. **(8+8)**

4. a) Show that for each positive integer $n \geq 1$. $n = \sum_{d/n} \phi(d)$, the sum being extended over all positive divisors of n.
   b) Explain the Hill Cipher with an example.
   c) Show that Dirichlet multiplication is commulative and associative. **(6+5+5)**

5. a) If 5 is a primitive root modulo 54, then find the remaining incongruent primitive roots.
   b) Solve the congruence : $x^3 \equiv 5 \pmod{13}$.
   c) Show that there exists a primitive root for p^k when p is an odd prime and $k \geq 1$. **(5+5+6)**

P.T.O.

6. a) Let p be an odd prime and gcd (a, p) = 1. Then show that 'a' is a quadratic residue of p if and only if $a^{(p-1)/2} \equiv 1 \pmod p$.

   b) State and prove the Gauss lemma for quadratic residue. Evaluate n of Gauss lemma for (11/23). **(6+10)**

7. a) Show that an odd prime p is expressible as a sum of two squares if and only if $p \equiv 1 \pmod 4$.

   b) Express 317 as sum of two squares.

   c) Show that $F_n^2 = F_{n+1} F_{n-1} + (-1)^{n-1}$. **(6+5+5)**

8. a) Define the finite continued fraction. Show that every rational number can be written as a finite simple continued fraction.

   b) Solve each linear diophantine equation using continued fraction

      i) $12x + 13y = 14$

      ii) $28x + 91y = 119$.

   c) If the number x has a periodic simple continued fraction expansion, then show that x is a quadratic irrational. **(5+6+5)**
