31$^{\text{st}}$ Indian National Mathematical Olympiad-2016

Time: 4 hours **January 17, 2016**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions. Maximum marks: 100.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be triangle in which $AB = AC$. Suppose the orthocentre of the triangle lies on the incircle. Find the ratio AB/BC.

2. For positive real numbers a, b, c, which of the following statements necessarily implies $a = b = c$: (I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$, (II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$? Justify your answer.

3. Let $\mathbb{N}$ denote the set of all natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k+1) = 2k+2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

   (i) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

   (ii) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

4. Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number $n \geq 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.

5. Let ABC be a right-angled triangle with $\angle B = 90°$. Let D be a point on AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of triangle ABC, prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

6. Consider a nonconstant arithmetic progression $a_1, a_2, \ldots, a_n, \ldots$. Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.

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