

Full Name : _____

Reference Code : _____

TATA INSTITUTE OF FUNDAMENTAL RESEARCH**Written Test in MATHEMATICS
December 13, 2009****Duration : Two hours (2 hours)****Please read all instructions carefully before you attempt the questions.**

1. Write your FULL NAME and REFERENCE CODE in block letters on this page and also fill-in all details on the ANSWER SHEET.
2. The Answer Sheet is machine-readable. Please read the instructions on the reverse of the answer sheet before you start filling it up. Only use HB pencils to fill-in the answer sheet.
3. There are thirty (30) questions divided into **TWO** parts (Part A & B) of 15 questions each. If you wish to be considered **ONLY** for the Integrated Ph.D. programme at Bengaluru, you need to answer only **Part A**. **All other candidates need to answer both Parts.**
4. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. Each correct answer will get 1 mark; each wrong answer will get a -1 mark, and a question not answered will not get you any mark. **Do no mark more than one circle for any question** : this will be treated as a wrong answer.
5. We advise you to first mark the correct answers in the QUESTION SHEET and then TRANSFER these to the ANSWER SHEET only when you are sure of your choice.
6. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator. Please do not scribble or do rough work on the reverse of your hall ticket. If found, the hall ticket will be retained.
7. **Use of calculators is NOT permitted.**
8. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.
9. **This set of question paper must be returned along with your answer sheet and all extra rough sheets used.**
10. See the back of this page for Notation and Conventions used in this test.

Notation and Conventions

\mathbb{Z} = set of integers

\mathbb{N} = set of natural numbers

\mathbb{Q} = set of rational numbers

\mathbb{R} = set of real numbers

\mathbb{C} = set of complex numbers

\mathbb{R}^n = Euclidean space of dimension n

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for real numbers a and b with $a < b$.

For a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$, f' denotes its derivative.

For any natural number n , $\mathbb{Z}/n\mathbb{Z}$ denotes the ring of integers modulo n .

Subsets of \mathbb{R}^n are assumed to carry the induced topology and metric.

Part A

1. A cyclic group of order 60 has
 - (a) 12 generators.
 - (b) 15 generators.
 - (c) 16 generators.
 - (d) 20 generators.
2. Which of the following is false?
 - (a) Any abelian group of order 27 is cyclic.
 - (b) Any abelian group of order 14 is cyclic.
 - (c) Any abelian group of order 21 is cyclic.
 - (d) Any abelian group of order 30 is cyclic.

3. The last digit of 2^{80} is

- (a) 2
- (b) 4
- (c) 6
- (d) 8

4. The sum of the series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{100.101}$$

is

- (a) $\frac{99}{101}$
- (b) $\frac{98}{101}$
- (c) $\frac{99}{100}$
- (d) None of the above.

5. Let f be an one to one function from the closed interval $[0, 1]$ to the set of real numbers \mathbb{R} , then

- (a) f must be onto.
- (b) range of f must contain a rational number.
- (c) range of f must contain an irrational number.
- (d) range of f must contain both rational and irrational numbers.

6. The maximum value of $f(x) = x^n(1-x)^n$ for a natural number $n \geq 1$ and $0 \leq x \leq 1$ is
- (a) $\frac{1}{2^n}$
 - (b) $\frac{1}{3^n}$
 - (c) $\frac{1}{5^n}$
 - (d) $\frac{1}{4^n}$
7. The sequence $\sqrt{7}, \sqrt{7 + \sqrt{7}}, \sqrt{7 + \sqrt{7 + \sqrt{7}}}, \dots$ converges to
- (a) $\frac{1 + \sqrt{33}}{2}$
 - (b) $\frac{1 + \sqrt{32}}{2}$
 - (c) $\frac{1 + \sqrt{30}}{2}$
 - (d) $\frac{1 + \sqrt{29}}{2}$
8. Let $f(x) = |x|^{\frac{3}{2}}$, $x \in \mathbb{R}$. Then
- (a) f is uniformly continuous.
 - (b) f is continuous, but not differentiable at $x = 0$.
 - (c) f is differentiable and f' is continuous.
 - (d) f is differentiable, but f' is discontinuous at $x = 0$.
9. The total number of subsets of a set of 6 elements is
- (a) 720
 - (b) 6^6
 - (c) 21
 - (d) None of the above.
10. Let $M_n(\mathbb{R})$ be the set of $n \times n$ -matrices with real entries. Which of the following statements is true?
- (a) Any matrix $A \in M_4(\mathbb{R})$ has a real eigenvalue.
 - (b) Any matrix $A \in M_5(\mathbb{R})$ has a real eigenvalue.
 - (c) Any matrix $A \in M_2(\mathbb{R})$ has a real eigenvalue.
 - (d) None of the above.

11. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

- (a) converges but not absolutely.
- (b) converges absolutely.
- (c) diverges.
- (d) none of the above.

12. The sum of the roots of the equation $x^5 + 3x^2 + 7 = 0$ is

- (a) -3
- (b) $\frac{3}{7}$
- (c) $\frac{-1}{7}$
- (d) 0

13. $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$ is

- (a) 1
- (b) 0
- (c) $\frac{1}{2}$
- (d) does not exist.

14. The solution of the ordinary differential equation

$$\frac{dy}{dx} = y, \quad y(0) = 0$$

- (a) is unbounded.
- (b) is positive.
- (c) is negative.
- (d) is zero.

15. Let G be the set of all 2×2 symmetric, invertible matrices with real entries. Then with matrix multiplication, G is

- (a) an infinite group.
- (b) a finite group.
- (c) not a group.
- (d) an abelian group.

Part B

- Let $u_n = \sin\left(\frac{\pi}{n}\right)$ and consider the series $\sum u_n$. Which of the following statements is false?
 - $\sum u_n$ is convergent.
 - $u_n \rightarrow 0$ as $n \rightarrow \infty$.
 - $\sum u_n$ is divergent.
 - $\sum u_n$ is absolutely convergent.
- If V is a vector space over the field $\mathbb{Z}/5\mathbb{Z}$ and $\dim_{\mathbb{Z}/5\mathbb{Z}}(V) = 3$, then V has
 - 125 elements.
 - 15 elements.
 - 243 elements.
 - None of the above.
- If $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are uniformly continuous functions, then their composition $g \circ f$ is
 - uniformly continuous.
 - continuous but not uniformly continuous.
 - continuous and bounded.
 - None of the above.
- Which of the following statements is false?
 - There exists a natural number which when divided by 3 leaves remainder 1 and which when divided by 4 leaves remainder 0.
 - There exists a natural number which when divided by 6 leaves remainder 2 and when divided by 9 leaves remainder 1.
 - There exists a natural number which when divided by 7 leaves remainder 1 and when divided by 11 leaves remainder 3.
 - There exists a natural number which when divided by 12 leaves remainder 7 and when divided by 8 leaves remainder 3.
- If $f_n(x)$ are continuous functions from $[0, 1]$ to $[0, 1]$, and $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, then which of the following statements is true?
 - $f_n(x)$ converges to $f(x)$ uniformly on $[0, 1]$.
 - $f_n(x)$ converges to $f(x)$ uniformly on $(0, 1)$.
 - $f(x)$ is continuous on $[0, 1]$.
 - None of the above.

6. Let A, B be subsets of \mathbb{R} . Define $A + B$ to be the set of all sums $x + y$ with $x \in A$ and $y \in B$. Which of the following statements is false?
- (a) If A and B are bounded, then $A + B$ is bounded.
 - (b) If A and B are open, then $A + B$ is open.
 - (c) If A and B are closed, then $A + B$ is closed.
 - (d) If A and B are connected, then $A + B$ is connected.

7. Number of solutions of the ordinary differential equation

$$\frac{d^2y}{dx^2} - y = 0, \quad y(0) = 0, \quad y(\pi) = 1$$

- (a) is 0.
 - (b) is 1.
 - (c) is 2.
 - (d) None of the above.
8. The function $f(x)$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

- (a) is not continuous at any point.
 - (b) is continuous at every point.
 - (c) is continuous at every rational number.
 - (d) is continuous at $x = 0$.
9. Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some positive integer } n\}$. Then under multiplication of complex numbers,
- (a) G is a group of finite order.
 - (b) G is a group of infinite order, but every element of G has finite order.
 - (c) G is a cyclic group.
 - (d) None of the above.
10. Let \mathbf{x} and \mathbf{y} in \mathbb{R}^n be non-zero column vectors. Form the matrix $A = \mathbf{x}\mathbf{y}^t$, where \mathbf{y}^t is the transpose of \mathbf{y} . Then the rank of A is
- (a) 2
 - (b) 0
 - (c) at least $n/2$.
 - (d) None of the above.

11. Which of the following is true?
- (a) The matrix $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ is not diagonalisable.
 - (b) The matrix $\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$ is diagonalisable.
 - (c) The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonalisable.
 - (d) None of the above.
12. If n and m are positive integers and $n^9 = 19m + r$, then the possible values for r modulo 19 are
- (a) only 0.
 - (b) only 0, ± 1 .
 - (c) only ± 1 .
 - (d) None of the above.
13. Define $\{x_n\}$ as $x_1 = 0.1$, $x_2 = 0.101$, $x_3 = 0.101001$, Then the sequence $\{x_n\}$
- (a) converges to a rational number.
 - (b) converges to an irrational number.
 - (c) does not converge.
 - (d) oscillates.
14. The equations
- $$x_1 + 2x_2 + 3x_3 = 1$$
- $$x_1 + 4x_2 + 9x_3 = 1$$
- $$x_1 + 8x_2 + 27x_3 = 1$$
- have
- (a) only one solution.
 - (b) two solutions.
 - (c) infinitely many solutions.
 - (d) no solution.
15. Which of the following statements is false?
- (a) The polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}/2\mathbb{Z}[x]$.
 - (b) The polynomial $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$.
 - (c) The polynomial $x^2 + 1$ is reducible in $\mathbb{Z}/5\mathbb{Z}[x]$.
 - (d) The polynomial $x^2 + 1$ is reducible in $\mathbb{Z}/7\mathbb{Z}[x]$.