IEOR @ IIT BOMBAY

PhD Admissions Screening Test, 09 December 2010, 9.30AM - 10.15AM

Name:
Category (GN/OBC-NC/SC/ST):
University:
Application Number:
Qualifying Degree:

Instructions: No clarifications on questions should be sought during the examination. Answer as many questions as you can.
Multiple choice questions: Write the most appropriate choice (example: (Q)) clearly in the space provided on the right side. Each correct answer gets two points (2) and wrong choice gets one negative point (-1).

1. Suppose a fair coin is tossed repeatedly and heads appeared in the first 1000 tosses. What is the probability that tail will turn up in the 1001th toss?
   (P) 2^{-1001} (Q) 0.5 (R) 0 (S) 1 Ans. ____

2. Consider a square of side 1 metre. A number of ants are dropped on the edges of the square at time t = 0. Once an ant is dropped, it starts walking along the edge (assume that the ants walk only along the edges) with a speed of 1 metre per minute. Once it reaches a corner, it falls off the square. If two ants collide head on, they immediately reverse directions and continue walking with the same speed. The earliest time after which one can be sure that all ants fall off the square is
   (P) 1 min (Q) 3 min (R) $\sqrt{2}$ min (S) never Ans. ____

3. Laila and Majnu go to a dinner party with four other couples. Each person there shakes hands with everyone he or she doesn't know. Later, Majnu does a survey and discovers that every one of the nine other attendees shook hands with a different number of people. How many people did Laila shake hands with?
   (P) 8 (Q) 1 (R) 7 (S) 4 Ans. ____

4. Five people of different heights go to a restaurant and sit at a round table at random. The probability that they are sitting in the order of their heights is
   (P) $\frac{1}{120}$ (Q) $\frac{1}{12}$ (R) $\frac{1}{2}$ (S) $\frac{1}{10}$ Ans. ____

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function. Suppose that x^* is a local minimum of f. Which of the following is true always?
   (P) $f''(x^*) > 0$ (Q) $f'(x^*) > 0$ (R) $f''(x^*) < 0$ (S) None of the above Ans. ____

6. Suppose that there are doors numbered 1 to 100 in a hallway. One person comes and opens all doors. A second person comes and closes all even numbered doors. Then a third person comes and changes the state of all doors that are multiples of 3. This continues until 100 people have passed the hallway. Which of the following doors remain open eventually?

(P) Door no. 50 (Q) Door no. 36 (R) Door no. 13 (S) None of the above ______ Ans.

7. Suppose a machine produces bolts, 10% of which are defective. What is the probability that a box of 3 bolts contains at most one defective bolt

(P) 1 (Q) 0.872 (R) 0.972 (S) $\frac{1}{4}$ ______ Ans.

8. Suppose a circle with centre at O and a line l intersects at a point P and nowhere else. Then the angle between OP and l is

(P) can not be determined from given data (Q) 180° (R) 90° (S) 45° ______ Ans.

9. When we write 23.723723723723..., we are writing a representation of

(P) an irrational number (Q) a nonterminating, nonrepeating decimal
(R) a rational number (S) a binary number ______ Ans.

10. Following table (Table 1) shows the results of a hypothetical experiment in which a 6-sided die is tossed 6000 times. What, if anything, is wrong with this table?

Face of die	Absolute frequency	Cumulative absolute frequency
1	1000	1000
2	1000	2000
3	1000	3000
4	1000	4000
5	1000	5000
6	1000	6000

Table 1: Results of a die experiment

(P) There is nothing wrong with the table; it is entirely plausible.
(Q) The data for the absolute frequency and the cumulative absoute frequency are in the wrong columns.
(R) There is no way a coincidence like this could ever occur
(S) The cumulative absolute frequency values do not add up right. ______ Ans.

11. Let $f : \mathbb{R} \to \mathbb{R}^2$ be a linear function which is not constant. What is the number of zeros of f?
(P) 1 (Q) 2 (R) 0 (S) infinite Ans. ____

12. Suppose you are sampling data from a Normal distribution with unknown mean. You have good estimates of the mean and standard deviation of the sample data. Which of the following can be determined on the basis of this information?
(P) The 50% confidence interval of the true mean.
(Q) The 90% confidence interval of the true mean.
(R) Both the above.
(S) None of the above. Ans. ____

13. Suppose the average weight of a given set of 100 persons is 60 kg, and their average height is 160 cm. Consider the three statements below:

(i) There must exist at least one person weighing at most 60 kg.

(ii) There must exist at least one person weighing at most 60 kg and having height at most 160 cm.

(iii) There must exist at least one person weighing at most 120 kg and having height at most 320 cm.

Which of the four alternatives below are true?
(P) Statement (i) is true and the other statements are false.
(Q) Statements (ii) and (iii) are true, and (i) is false.
(R) Statement (iii) is true, and the others are false.
(S) Statements (i) and (iii) are true, and (ii) is false. Ans. ____

14. Consider the algorithm below.

$f(x) = (3 - x)^2$
$x_0 = 0$
for (k = 0; k $\leq$ 10; k++)
{
  $d(x_k) = f'(x_k)$
  $dd(x_k) = d'(x_k)$
  $x_{k+1} = x_k - \frac{d(x_k)}{dd(x_k)}$
}

On execution, the value of $x_2 =$
(P) -2 (Q) -3 (R) 2 (S) 3 Ans. ____

15. Two fair coins A and B, made of different materials, are tossed simultaneously. Due to differences in the material of the coins, coin A lands first in 90% of the experiments. If coin A lands first, the coin B always shows the same face as coin A. Else, the outcomes are independent. What is the probability that both coins show the same face? Ans. ____

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PhD Admissions Test, 09 December 2010, 10.15AM - 12.15PM

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Instructions: Answer at least one question from each of Part A, Part B and Part C. Please give all details of your work. No clarifications on questions should be sought during the examination. You can make any assumptions if you want, but you need to state them clearly.

Part A

A1 Consider the function $L(x) : \mathbb{R} \to \mathbb{R}$ defined as $L(x) := \max\{a_1 x + b_1, \ldots, a_n x + b_n\}$ for a given $2n$ reals $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$.

1. Is $L(\cdot)$ a continuous function?
2. Is $L(\cdot)$ a differentiable function?
3. Give sufficient conditions under which it is a differentiable function. Are these conditions also necessary?
4. A real valued function $f(\cdot)$ on real line $\mathbb{R}$ is said to be convex if for any two given reals x and y, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $\lambda \in (0,1)$. Is $L(\cdot)$ a convex function?
5. A function $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is said to be a concave function if $-f(\cdot)$ is a convex function. Is $L(\cdot)$ a concave function?
6. Give sufficient conditions under which $L(\cdot)$ is a concave function.
7. Give sufficient conditions under which $L(\cdot)$ is both a convex and a concave function.
8. A set $A \subseteq \mathbb{R}^2$ is a convex set if for any given pair of points $x,\ y\ \in A$, we have point $\lambda x + (1-\lambda)y\ \in A$ for all $\lambda\ \in (0,1)$. Identify a convex set in $\mathbb{R}^2$, if any, when function $L(\cdot)$ is plotted.

A2

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Let $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ denote the standard basis vectors for $\mathbb{R}^3$. In addition, we know that
$$T\mathbf{e}_1 = \begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix},\ T\mathbf{e}_2 = \begin{pmatrix} 8 \\ 1 \\ -4 \end{pmatrix},\ \text{and } T\mathbf{e}_3 = \begin{pmatrix} 16 \\ 8 \\ -11 \end{pmatrix}.$$
Determine the matrix A that defines the linear transformation T.
2. Compute the eigenvalues of the matrix A (that you determined previously in part 1).
3. Compute the eigenvectors corresponding to the eigenvalues you computed in part 2.
4. Consider the composite linear map $T \circ T : \mathbb{R}^3 \to \mathbb{R}^3$, defined as $T \circ T(x) = T(T(x))$. Determine the matrix B that defines $T \circ T$.
5. Compute the eigenvalues of B and the corresponding eigenvectors.

A3

1. Let C be the 100×100 matrix of real numbers whose entry in the (i,j)th place is ij for $1 \leq i,j \leq 100$. For example, $c_{1,2} = 1 \cdot 2 = 2$, and $c_{5,50} = 5 \cdot 50 = 250$. Compute the rank of the matrix C.
2. Let D be a $n \times n$ matrix, and let $x \in \mathbb{R}^n$. Show that there exists a positive integer m such that $D^m x = \mathbf{0}$, but $D^{m-1}x \neq \mathbf{0}$.
3. Let M be a 7×7 matrix with real entries such that $M^2 = \mathbf{0}_{7\times 7}$, the zero matrix. What are the possible values of the nullity of M? Don't just list some numbers, justify your answers!

Part B

B1 Let $A_1, A_2, \ldots, A_n$ be independent events defined on a sample space Ω such that $0 < \mathbb{P}(A_j) < 1$ for $j = 1, \ldots, n$ (here, $\mathbb{P}(A_j)$ denotes the probability that event A_j occurs). Then show that Ω must contain at least 2^n events.

B2 Let X be uniformly distributed over the interval $[-1, 1]$. Let Y be defined as $Y = X^2$.

(a) Are X and Y independent random variables?

(b) Compute $\mathbb{E}[XY]$, the expected value of XY. Also, compute $\mathbb{E}[X]\mathbb{E}[Y]$, and verify if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

(c) Comment on the above observations.

B3 Let `random()` be a function that takes an integer $n \geq 1$ as input, and outputs a random integer from $\{1, \ldots, n\}$. Consider the following pseudo-code:

```
x = 0;
do while (N > 1)
    N = random(N);
    x = x + 1;
end do
```

Let $\mathbb{E}(n)$ denote the *expected value* of the result of the above pseudocode with initialization $N = n$.

1. Compute the values of $\mathbb{E}(1)$ and $\mathbb{E}(2)$.
2. Derive a recurrence relation that will help you compute $\mathbb{E}(n)$ when $\mathbb{E}(1), \ldots, \mathbb{E}(n-1)$ are given.
3. Compute $\mathbb{E}(n) - \mathbb{E}(n-1)$.
4. Using part 3, determine a closed-form formula for $\mathbb{E}(n)$.

Part C

Orange Computer Inc., is planning to launch a series of laptop computers into the market called "jPad". Orange is well-aware of the segmentation of the market into different age groups and household income levels in terms of their willingness to pay for the computer. In an attempt to capture as much revenue as possible, Orange decided to release four variants of the jPad (inspira, lifepad, nano, futura) at different prices. After a careful market survey, Orange determined a detailed characterization of the market. Let N denote the different segments of the market (these segments are different types of customers), and let n_i denote the size (i.e., the number of customers) of segment $i \in N$. Let J denote the set of different jPads, and let r_{ij} denote the maximum price market segment i is willing to pay for jPad $j \in J$.

Economic theory suggests that a customer will buy the product that gives the greatest non-negative consumer surplus, where consumer surplus is defined as the maximum price the customer is willing to pay minus the actual price of the object. For instance, if Orange decides to set the prices for the jPads (in thousands of Rupees) as 80, 90, 90, 100 for inspira, lifepad, nano, and futura respectively, and a particular segment is willing to pay 70, 90, 120, and 120 respectively, the consumer surpluses are -10, 0, 30, and 20 respectively, and this segment will buy the jPad nano as this product gives the largest consumer surplus. In the case of all consumer surpluses being less than or equal to 0, customers will not buy any jPad.

Our objective is to write an optimization model to determine what prices should Orange set for the jPads in order to maximize revenue from sales. Assume that a customer will buy at most one jPad. Let us use the following decision variables.

p_j	The price Orange sets for jPad $j \in J$
s_i	The surplus of customer $i \in N$
x_{ij}	1 if customer i buys jPad j, and 0 otherwise.

1. Consider the following *nonlinear* optimization problem:

$$
\text{(NL)} \qquad
\begin{array}{llrcll}
\max & & \displaystyle\sum_{i \in N} \sum_{j \in J} n_i p_j x_{ij} & & & (1) \\
\text{s. t.} & & s_i - r_{ij} + p_j & \geq & 0 & (2) \\
 & & \displaystyle s_i - \sum_{j \in J} (r_{ij} - p_j) x_{ij} & = & 0 & (3) \\
 & & s_i, p_j & \geq & 0, \quad i \in N,\ j \in J & (4) \\
 & & x_{ij} & \in & \{0,1\}, \quad i \in N,\ j \in J. & (5)
\end{array}
$$

   Explain the significance of the objective function and explain the meaning of each constraint. Does the optimization problem (NL) model Orange's problem correctly? If so, why? If not, how can you modify the model so that it models the situation correctly?

2. The optimization model (NL) is nonlinear, as it involves products of variables. Introduce a set of variables z_{ij}, where $i \in N$, and $j \in J$ and write a mixed-integer program with **linear objective** and **linear constraints** that is equivalent to the model (NL).