Ph. D Written Examination, May 9th 2016. Time 2 hrs, Marks 100.

Please Answer ALL Questions SERIALLY.

1) The following nonlinear oscillator

$$\ddot{x} + \beta x^3 = 0\,; \qquad \beta \in \mathbb{R}^+,$$

models the one dimensional motion of a point particle of unit mass moving under the influence of a nonlinear restoring force $-\beta x^3$. The system is being observed in an inertial frame F with coordinates (x, t).

(a) Write down the Lagrangian for this mechanical system. [4]

(b) Find out the time-period of the oscillatory states in terms of $\Gamma(1/4)$, β, and the total energy (E). Use arbitrary initial conditions. [*Hint:* $\Gamma(x) =: \int_0^\infty t^{x-1}\exp(-t)dt$, $\mathrm{B(p,q)} =: \int_0^1 t^{p-1}(1-t)^{q-1}dt$, $\mathrm{B(p,q)} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, and $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$.] [10]

(c)Suppose the system is being observed from a (non-relativistic) frame F' (with coordinates (x', t')) accelerating with a constant acceleration α w.r.t. F and moving in positive x direction. What is the Lagrangian for the system in frame F'? [6]

2) Consider a two-level atom with states $|e\rangle$ and $|g\rangle$. The time-independent Hamiltonian governing this system is

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{int}$$

Here:

$$\hat{H}_{atom} = \hbar\Delta\,|e\rangle\,\langle e|\ \ ,\ \hat{H}_{int} = -\frac{\hbar\Omega}{2}(|e\rangle\,\langle g| + |g\rangle\,\langle e|),$$

where $\hbar\Omega = \langle e|\vec{d}|g\rangle$ and $\Delta = (\omega_0 - \omega_l)$, with $\vec{d}$ being the electric dipole moment, ω_0 the frequency separation between the two atomic levels and ω_l the frequency of the field.

(a) Find the energy eigenvalues and eigenvectors of the system. [4]

(b) Plot the eigenenergies as a function of Δ in presence and absence of interaction. [4]

(c) Point out the differences when $\Delta < 0$ and $\Delta > 0$. [2]

(d) Expand the solution (a) to lowest nonvanishing order in $\frac{\Omega}{\Delta}$. [2]

(e) Given that $|\psi(t=0)\rangle = |e\rangle\,\langle e|$, explicitly obtain its time evolution. Obtain the probability for the system being in the excited state $|e\rangle$, and plot it as a function of time. [4+4]

3a) Consider a sphere of radius R having a uniform volume charge density ρ. Calculate the electric field $\mathbf{E}(\mathbf{r})$ due to the sphere everywhere. [4]

3b) A point charge q is at the center of an uncharged spherical conducting shell, of inner radius a and outer radius b. Calculate how much work it would take to move the charge out to infinity (through a tiny hole drilled in the shell)? [8]

3c) Show that in a region of vacuum that is free of charges and currents the electric field ($\mathbf{E}$) follows the following wave equation: $\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$, where c is the speed of light in vacuum. [2]

3d) Show that the plane wave $\mathbf{E}(\mathbf{r}, t) = E_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ is a solution to the wave equation for the electric field, with $c = \omega/|\mathbf{k}|$ being the speed of light in vacuum. [3]

3e) A plane wave has an infinite spatial extent. Using the fact that a plane wave is a solution to the wave equation, construct a solution to the wave equation that has a finite spatial extent in the $x - y$ plane at $z = 0$. [3]

4) Consider a localized spin-1/2 in a uniform magnetic field B applied in the z-direction at a temperature T.

a) The Hamiltonian $H = -\mu_B B S_z$; here μ_B is the Bohr magneton and S_z is a binary variable taking values ± 1. What is the canonical partition function? Find the average $\langle S_z \rangle$. [3+2]

b) If the Hamiltonian is $H = -\mu_B B \hat{\sigma}_z$, where $\hat{\sigma}_z$ is the Pauli spin. Write down the canonical density matrix and calculate $\langle \hat{\sigma}_z \rangle$ [3+2]

5) Evaluate the following integral (with $a > b$)

$$I = \int_0^\pi \frac{d\theta}{a - b\cos\theta}.$$

[10]