

NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 22, 2011

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit.**
- \mathbb{N} denotes the set of natural numbers, \mathbb{Z} - the integers, \mathbb{Q} - the rationals, \mathbb{R} - the reals and \mathbb{C} - the field of complex numbers. \mathbb{R}^n denotes the n -dimensional Euclidean space, which is assumed to be endowed with its ‘usual’ topology. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual ‘sup’ norm.

Section 1: Algebra

1.1 Solve:

$$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0.$$

1.2 Pick out the true statements:

a. Let H and K be subgroups of a group G . For $g \in G$, define the double coset

$$HgK = \{h g k \mid h \in H, k \in K\}.$$

Then, if H is normal, we have $HgH = gH$ for all $g \in G$.

b. Let $GL(n; \mathbb{C})$ be the group of all $n \times n$ invertible matrices with complex entries. The set of all $n \times n$ invertible upper triangular matrices is a normal subgroup.

c. Let $\mathbb{M}(n; \mathbb{R})$ denote the set of all $n \times n$ matrices with real entries (identified with \mathbb{R}^{n^2} and endowed with its usual topology) and let $GL(n; \mathbb{R})$ denote the group of invertible matrices. Let G be a subgroup of $GL(n; \mathbb{R})$. Define

$$H = \left\{ A \in G \mid \begin{array}{l} \text{there exists } \varphi : [0, 1] \rightarrow G \text{ continuous,} \\ \text{such that } \varphi(0) = A, \varphi(1) = I \end{array} \right\}.$$

Then, H is a normal subgroup of G .

1.3 How many (non-isomorphic) groups of order 15 are there?

1.4 Pick out the true statements:

a. Let R be a commutative ring with identity. Let M be an ideal such that every element of R not in M is a unit. Then R/M is a field.

b. Let R be as above and let M be an ideal such that R/M is an integral domain. Then M is a prime ideal.

c. Let $R = \mathcal{C}[0, 1]$ be the ring of real-valued continuous functions on $[0, 1]$ with respect to pointwise addition and pointwise multiplication. Let

$$M = \{f \in R \mid f(0) = f(1) = 0\}.$$

Then M is a maximal ideal.

1.5 Write down all the possible values for the degree of an irreducible polynomial in $\mathbb{R}[x]$.

1.6 Let V be the real vector space of all polynomials in $\mathbb{R}[x]$ with degree less than, or equal to 4. Consider the linear transformation which maps $p \in V$ to its derivative p' . If the matrix of this transformation with respect to the basis $\{1, x, x^2, x^3, x^4\}$ is A , write down the matrix A^3 .

1.7 Let $\mathbb{T}(n; \mathbb{R}) \subset \mathbb{M}(n; \mathbb{R})$ denote the set of all matrices whose trace is zero. Write down a basis for $\mathbb{T}(2; \mathbb{R})$.

1.8 What is the quotient space $\mathbb{M}(n; \mathbb{R})/\mathbb{T}(n; \mathbb{R})$ isomorphic to?

1.9 Construct a 2×2 matrix $A (\neq I)$ with real entries such that $A^3 = I$.

1.10 If $A \in \mathbb{M}(n; \mathbb{R})$, let tA denote its transpose. A matrix $S \in \mathbb{M}(n; \mathbb{R})$ is said to be *skew-symmetric* if ${}^tS = -S$. Pick out the true statements:

- a. If $S \in \mathbb{M}(n; \mathbb{R})$ is skew-symmetric and non-singular, then n is even.
- b. Let

$$G = \{T \in GL(n; \mathbb{R}) \mid {}^tTST = S, \text{ for all skew-symmetric } S \in \mathbb{M}(n; \mathbb{R})\}.$$

Then G is a subgroup of $GL(n; \mathbb{R})$.

c. Let I_n and O_n denote the $n \times n$ identity and null matrices respectively. let S be the $2n \times 2n$ matrix given in block form by

$$\begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix}.$$

If X is a $2n \times 2n$ matrix such that ${}^tXS + SX = 0$, then the trace of X is zero.

Section 2: Analysis

2.1 Let $\{a_n\}$ be a sequence of positive terms. Pick out the cases which imply that $\sum a_n$ is convergent.

a.

$$\lim_{n \rightarrow \infty} n^{\frac{3}{2}} a_n = \frac{3}{2}.$$

b.

$$\sum n^2 a_n^2 < \infty.$$

c.

$$\frac{a_{n+1}}{a_n} < \left(\frac{n}{n+1} \right)^2, \text{ for all } n.$$

2.2 Evaluate:

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{1+n^3} + \frac{4}{8+n^3} + \cdots + \frac{n^2}{n^3+n^3} \right\}.$$

2.3 Find the points in \mathbb{R} where the following function is differentiable:

$$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \leq 1 \\ \frac{\pi}{4} \operatorname{sgn}(x) + \frac{|x|-1}{2}, & \text{if } |x| > 1, \end{cases}$$

where $\operatorname{sgn}(x)$ equals $+1$ if $x > 0$, -1 if $x < 0$ and is equal to zero if $x = 0$ and $\tan^{-1}(x)$ takes its values in the range $]-\pi/2, \pi/2[$ for real numbers x .

2.4 Pick out the true statements:

a. If P is a polynomial in one variable with real coefficients which has all its roots real, then its derivative P' has all its roots real as well.

b. The equation $\cos(\sin x) = x$ has exactly one solution in the interval $[0, \frac{\pi}{2}]$.

c. $\cos x > 1 - \frac{x^2}{2}$ for all $x > 0$.

2.5 Let $f, f_n : [0, 1] \rightarrow \mathbb{R}$ be continuous functions. Complete the following sentence such that both statements (a) and (b) below are true:

“Let $f_n \rightarrow f$ ”

a.

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx.$$

b.

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow 0} f_n(x) = \lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} f_n(x).$$

2.6 Let $f :]0, 1[\rightarrow \mathbb{R}$ be continuous. Pick out the statements which imply that f is uniformly continuous.

a. $|f(x) - f(y)| \leq \sqrt{|x - y|}$, for all $x, y \in]0, 1[$.

b. $f(1/n) \rightarrow 1/2$ and $f(1/n^2) \rightarrow 1/4$.

c.

$$f(x) = x^{\frac{1}{2}} \sin \frac{1}{x^3}.$$

2.7 Evaluate:

$$\int \int_{[0,1] \times [0,1]} \max\{x, y\} \, dx dy.$$

2.8 Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be analytic. Pick out the cases when f is **not** necessarily a constant.

- a. $\operatorname{Im}(f'(z)) > 0$ for all $z \in \mathbb{C}$.
- b. $f(n) = 3$ for all $n \in \mathbb{Z}$.
- c. $f'(0) = 0$ and $|f'(z)| \leq 3$ for all $z \in \mathbb{C}$.

2.9 Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be analytic. Write $z = x + iy$ and $f = u + iv$, where u and v are real valued functions of x and y . Pick out the true statements.

a.

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

b.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

c.

$$f''(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^3} \, dz.$$

2.10 Find the square roots of $1 + i\sqrt{3}$.

Section 3: Topology

3.1 Which of the following define a metric?

- $d((x, y), (x', y')) = \min\{|x - x'|, |y - y'|\}$ on \mathbb{R}^2 .
- $d((x, y), (x', y')) = |x| + |y| + |x'| + |y'|$ on \mathbb{R}^2 .
- $D((x, y), (x', y')) = d(x, x') + d(y, y')$ on $X \times X$, where (X, d) is a metric space.

3.2 Let (X, d) be a metric space and let $A \subset X$. For $x \in X$ define

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

Pick out the true statements:

- $x \mapsto d(x, A)$ is a uniformly continuous function.
- If

$$\partial A = \{x \in X \mid d(x, A) = 0\} \cap \{x \in X \mid d(x, X \setminus A) = 0\},$$

then ∂A is closed for any $A \subset X$.

- Let A and B be subsets of X and define

$$d(A, B) = \inf\{d(a, B) \mid a \in A\}.$$

Then $d(A, B) = d(B, A)$.

3.3 Let X be a topological space and for $A \subset X$, denote by \overline{A} and A° , the closure and interior of A respectively. Pick out the true statements.

- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
- $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
- Consider \mathbb{R} as the x -axis in \mathbb{R}^2 . Then $\mathbb{R}^\circ = \emptyset$.

3.4 Pick out the true statements.

- Let $\{X_i\}_{i \in \mathcal{I}}$ be topological spaces. Then, the product topology is the smallest topology on $X = \prod_{i \in \mathcal{I}} X_i$ such that each of the canonical projections $p_i : X \rightarrow X_i$ is continuous.
- Let X be a topological space and $W \subset X$. Then, the induced subspace topology on W is the smallest topology such that $id|_W : W \rightarrow X$, where id is the identity map, is continuous.
- Let $X = \mathbb{R}^n$ with the usual topology. This is the smallest topology such that all linear functionals on X are continuous.

3.5 Which of the following subsets are dense in the given spaces?

- The set of trigonometric polynomials in the space of continuous functions on $[-\pi, \pi]$ which are 2π -periodic (with the sup-norm topology).
- The subset of C^∞ functions with compact support in \mathbb{R} in the space of bounded real-valued continuous functions on \mathbb{R} (with the sup-norm topology).
- $GL(n; \mathbb{R})$ in $M(n; \mathbb{R})$ (with its usual topology after identification with \mathbb{R}^{n^2}).

3.6 Pick out the compact sets.

- $\{(x, y) \mid x^2 - y^2 = 1\} \subset \mathbb{R}^2$.
- $\{\text{Tr}(A) \mid A \in M(n; \mathbb{R}), A \text{ orthogonal}\} \subset \mathbb{R}$, where $\text{Tr}(A)$ denotes the trace of the matrix A .
- The set of all matrices in $M(n; \mathbb{R})$ all of whose eigenvalues satisfy the condition $|\lambda| \leq 2$.

3.7 Pick out the connected sets.

- $\{(x, y) \mid xy = 1\} \subset \mathbb{R}^2$.
- The set of all upper triangular matrices in $M(n; \mathbb{R})$.
- The set of all invertible diagonal matrices in $M(n; \mathbb{R})$.

3.8 Pick out the true statements.

- Let $f : \mathbb{Z} \rightarrow \mathbb{Z}^2$ be a bijection. There exists a continuous function from \mathbb{R} to \mathbb{R}^2 which extends f .
- Let D denote the closed unit disc in \mathbb{R}^2 . There exists a continuous mapping $f : D \setminus \{(0, 0)\} \rightarrow \{x \in \mathbb{R} \mid |x| \leq 1\}$ which is onto.
- Let D denote the closed unit disc in \mathbb{R}^2 . There exists a continuous mapping $f : D \setminus \{(0, 0)\} \rightarrow \{x \in \mathbb{R} \mid |x| > 1\}$ which is onto.

3.9 A Hausdorff topological space is said to be normal if given any two disjoint closed sets A and B , there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Pick out the true statements.

- Every metric space is normal.
- If X is a normal space with at least two distinct points, then there exist non-constant real-valued continuous functions on X .
- If X is normal and $Y \subset X$ is closed, then Y is normal for the induced topology.

3.10 Which of the following pairs of sets are homeomorphic?

- $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 = 0\}$ and $B = \{(x, y) \mid 5x^2 + 3y^2 = 1\}$.
- $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 = 0\}$ and $B = \{(x, y) \mid 5x^2 - 3y^2 = 1\}$.
- $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 \leq 0\}$ and $B = \{(x, y) \mid 5x^2 + 3y^2 \geq 1\}$.

Section 4: Applied Mathematics

4.1 Simpson's rule is used to approximate the integral $\int_0^1 f(x) dx$. If f is a polynomial, what is the maximum possible degree it can have so that Simpson's rule gives the exact value of this integral?

4.2 A right circular cylinder of fixed volume has maximum total surface area. What is the relationship between its height h and radius r ?

4.3 In the equations governing the flow of an incompressible fluid of uniform density, if \mathbf{u} is the velocity vector and p is the pressure, write down the equation which expresses the law of conservation of mass.

4.4 A particle of mass M is attached to a fixed wall by a spring. The spring exerts no force when the particle is at its equilibrium position at $x = 0$ and exerts a restoring force proportional to the displacement when it is displaced to a distance x . In addition, there is a damping force due to the medium in which the displacement takes place, which is a force opposing the motion and is proportional to the velocity of the particle. If the particle is pulled to a position x_0 at time $t = 0$ and is released without any velocity, write down the initial value problem governing the motion of the particle.

4.5 Solve the following linear programming problem:

$$\begin{aligned}\max z &= 5x + 7y \\ x - y &\leq 1 \\ 2x + y &\geq 2 \\ x + 2y &\leq 4 \\ x, y &\geq 0.\end{aligned}$$

4.6 Write down the dual of the above problem.

4.7 Find the general solution of the system:

$$\begin{aligned}\frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 4x - 2y.\end{aligned}$$

4.8 Let \mathbf{a} , \mathbf{b} and \mathbf{c} be vectors in \mathbb{R}^3 . Express $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ as a linear combination of \mathbf{b} and \mathbf{c} .

4.9 Solve:

$$\frac{\partial^2 u}{\partial x^2} = 6xy; \quad u(0, y) = y; \quad \frac{\partial u}{\partial x}(1, y) = 0.$$

4.10 Let Ω be a smooth plane domain of unit area. Let $u(x, y) = 3x^2 + y^2$. If $\frac{\partial u}{\partial n}$ denotes its outer normal derivative on $\partial\Omega$, the boundary of Ω , compute

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} ds.$$

Section 5: Miscellaneous

5.1 Let V be a real vector space of real-valued functions on a given set. Assume that constant functions are in V and that if $f \in V$, then $f^2 \in V$ and that $|f| \in V$. Pick out the true statements.

- If $f, g \in V$, then $fg \in V$.
- If $f, g \in V$, then $\max\{f, g\} \in V$.
- If $f \in V$ and p is any polynomial in one variable, with real coefficients, then $p(f) \in V$.

5.2 A fair coin is tossed 10 times, the tosses being independent of each other. Find the probability that the results of the third, fourth and fifth tosses are identical.

5.3 Determine if the following collections are countable or uncountable.

- The collection of all finite subsets of \mathbb{N} .
- The collection of all infinite sequences of positive integers.
- The collection of all roots of all polynomials in one variable, with integer coefficients.

5.4 Find the maximum value of $x + 2y + 3z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

5.5 Let A_n be the $n \times n$ matrix whose (i, j) -th entry is given by

$$2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}$$

where δ_{ij} equals 1 if $i = j$ and zero otherwise. Compute the determinant of A_n .

5.6 How many real roots does the following equation have?

$$3^x + 4^x = 5^x$$

5.7 Let $N > 1$ be a positive integer. Let $\phi(N)$ denote the number of positive integers less than N and prime to it (unity being included in this count). Express the sum of all the integers less than N and prime to it in terms of $\phi(N)$.

5.8 Pick out the true statements.

- The sum of r consecutive positive integers is divisible by r .
- The product of r consecutive positive integers is divisible by $r!$.
- For each positive integer r , there exist r consecutive positive integers which are all composite.

5.9 Let n be a fixed positive integer and let $0 \leq k \leq n$. We denote by C_k , the number of ways of choosing k objects from n distinct objects. Sum to n terms:

$$3C_1 + 7C_2 + 11C_3 + \cdots$$

5.10 Find the sum of the following infinite series:

$$\frac{1}{5} - \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} - \cdots$$