

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)

Organized by
Department of Mathematics, S. P. College, Pune
and

Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 08/01/2017

Max. Marks: 100

Time: 12.00 noon to 3.00 p.m.

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number $\sqrt{2}e^{i\pi}$ is:
A) a rational number.
B) an irrational number.
C) a purely imaginary number.
D) a complex number of the type $a + ib$ where a, b are non-zero real numbers.
2. Let $P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The rank of P^4 is: A) 1 B) 2 C) 3 D) 4.
3. Let $y_1(x)$ and $y_2(x)$ be the solutions of the differentiable equation $\frac{dy}{dx} = y + 17$ with initial conditions $y_1(0) = 0, y_2(0) = 1$. Which of the following statements is true?
A) y_1 and y_2 will never intersect.
B) y_1 and y_2 will intersect at $x = e$.
C) y_1 and y_2 will intersect at $x = 17$.
D) y_1 and y_2 will intersect at $x = 1$.
4. Suppose f and g are differentiable functions and $h(x) = f(x)g(x)$. Let $h(1) = 24$, $g(1) = 6$, $f'(1) = -2$, $h'(1) = 20$. Then the value of $g'(1)$ is
A) 8 B) 4 C) 2 D) 16.
5. In how many regions is the plane divided when the following equations are graphed, not considering the axes? $y = x^2, y = 2^x$
A) 3 B) 4 C) 5 D) 6.
6. For $0 \leq x < 2\pi$, the number of solutions of the equation $\sin^2 x + 3 \sin x \cos x + 2 \cos^2 x = 0$ is
A) 1 B) 2 C) 3 D) 4.
7. The minimum value of the function $f(x) = x^x, x \in (0, \infty)$ is
A) $\left(\frac{1}{10}\right)^{\frac{1}{10}}$ B) $10^{\frac{1}{10}}$ C) $\frac{1}{e}$ D) $\left(\frac{1}{e}\right)^{\frac{1}{e}}$.
8. Let f be a twice differentiable function on $\mathbb{R}$. Also $f''(x) > 0$ for all $x \in \mathbb{R}$. Which of the following statements is true?
A) $f(x) = 0$ has exactly two solutions on $\mathbb{R}$.
B) $f(x) = 0$ has a positive solution if $f(0) = 0$ and $f'(0) = 0$.
C) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) > 0$.
D) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) < 0$.

9. If $x^2 + x + 1 = 0$, then the value of $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \dots + \left(x^{27} + \frac{1}{x^{27}}\right)^2$ is
 A) 27 B) 54 C) 0 D) -27.
10. Let $M = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$. Then $M^{2017} =$
 A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ B) $\begin{pmatrix} -2^{2017} & -1 \\ 3^{2017} & 1 \end{pmatrix}$ C) $\begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$ D) $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$.

Part II

N.B. Each question in Part II carries 6 marks.

- Let a, b, c be real numbers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $abc = 1$. Prove that at least one of a, b, c is 1.
- Let $c_1, c_2, \dots, c_9$ be the zeros of the polynomial $z^9 - 6z^7 + 12z^6 + 18z^4 - 24z^3 + 30z^2 - z + 2017$. If $S(z) = \sum_{k=1}^9 |z - c_k|^2$, then prove that $S(z)$ is constant on the circle $|z| = 100$.
- Let f be a monic polynomial with real coefficients. Let $\lim_{x \rightarrow \infty} f''(x) = \lim_{x \rightarrow \infty} f\left(\frac{1}{x}\right)$ and $f(x) \geq f(1)$ for all $x \in \mathbb{R}$. Find f .
- Call a set of integers *non-isolated* if for every $a \in A$ at least one of the numbers $a - 1$ and $a + 1$ also belongs to A . Prove that the number of 5-element *non-isolated* subsets of $\{1, 2, \dots, n\}$ is $(n - 4)^2$.
- Find all positive integers n for which a permutation $a_1, a_2, \dots, a_n$ of $\{1, 2, \dots, n\}$ can be found such that $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_n$ leave distinct remainders modulo $n + 1$.

Part III

- Do there exist 100 lines in the plane, no three concurrent such that they intersect exactly in 2017 points? [12]
- On the parabola $y = x^2$, a sequence of points $P_n(x_n, y_n)$ is selected recursively where the points P_1, P_2 are arbitrarily selected distinct points. Having selected P_n , tangents drawn at P_{n-1} and P_n meet at say Q_n . Suppose P_{n+1} is the point of intersection of $y = x^2$ and the line passing through Q_n parallel to Y-axis. Under what conditions on P_1, P_2
 - both the sequences $\{x_n\}$ and $\{y_n\}$ converge?
 - $\{x_n\}$ and $\{y_n\}$ both converge to 0? [13]
- Show that there does not exist a 3-digit number A such that $10^3 A + A$ is a perfect square.
 - Show that there exists an n -digit ($n > 3$) number A such that $10^n A + A$ is a perfect square. [12]
- For $n \times n$ matrices A, B , let $C = AB - BA$. If C commutes with both A and B , then
 - Show that $AB^k - B^k A = kB^{k-1}C$ for every positive integer k .
 - Show that there exists a positive integer m such that $C^m = 0$. [13]