Part I

N.B. Each question in Part I carries 2 marks.

1. If $N = 1! + 2! + 3! + \cdots + 2011!$, then the digit in the units place of the number N is
   (a) 1 (b) 3 (c) 0 (d) 9.

2. The set of all points z in the complex plane satisfying $z^2 = |z|^2$ is a
   (a) pair of points (b) circle (c) union of lines (d) line.

3. If the arithmetic mean of two numbers is 26 and their geometric mean is 10, then the equation with these two numbers as roots is
   (a) $x^2 + 52x + 100 = 0$ (b) $x^2 - 52x - 100 = 0$
   (c) $x^2 - 52x + 100 = 0$ (d) $x^2 + 52x - 10 = 0$.

4. All points lying inside the triangle with vertices at the points $(1, 3)$, $(5, 0)$ and $(-1, 2)$ satisfy
   (a) $3x + 2y \geq 0$ (b) $2x + y - 13 \geq 0$
   (c) $2x - 3y - 12 \geq 0$ (d) $-2x + y \geq 0$.

5. For $n \geq 3$, let A be an $n \times n$ matrix. If rank of A is $n - 2$, then rank of adjoint of A is
   (a) $n - 2$ (b) 2 (c) 1 (d) 0.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an odd and differentiable function. Then for every $x_0 \in \mathbb{R}$, $f'(-x_0)$ is equal to
   (a) $f'(x_0)$ (b) $-f'(x_0)$ (c) 0 (d) None of these.

7. If $S = \{a, b, c\}$ and the relation R on the set S is given by $R = \{(a, b), (c, c)\}$, then R is
   (a) reflexive and transitive (b) reflexive but not transitive
   (c) not reflexive but transitive (d) neither reflexive nor transitive.

8. Let $a_1 = 1,\ a_{n+1} = \left(\dfrac{1+n}{n}\right) a_n$ for $n \geq 1$. Then the sequence $\{a_n\}$ is
   (a) divergent (b) decreasing (c) convergent (d) bounded.

9. The coefficient of x^{2n-2} in $(x-1)(x+1)(x-2)(x+2)\cdots(x-n)(x+n)$ is
   (a) 0 (b) $\dfrac{-n(n+1)(2n+1)}{6}$ (c) $\dfrac{n(n+1)(2n+1)}{6}$ (d) $\dfrac{-n(n+1)}{2}$.

10. The number of roots of $5x^4 - 4x + 1 = 0$ in $[0, 1]$ is
    (a) 0 (b) 1 (c) 2 (d) 3.

Part II

N.B. Each question in Part II carries 5 marks.

1. If $n \geq 3$ is an integer and k is a real number, prove that n is equal to the sum of n^{th} powers of the roots of the equation $x^n - kx - 1 = 0$.

2. Find all positive integers n such that $(n2^n - 1)$ is divisible by 3.

3. Start with the set $S = \{3, 4, 12\}$. At any stage you may perform the following operation: Choose any two elements $a, b \in S$ and replace them by $\left(\frac{3a - 4b}{5}\right)$ and $\left(\frac{4a + 3b}{5}\right)$. Is it possible to transform the set S into the set $\{4, 6, 12\}$ by performing the above operation a finite number of times?

4. Let $a < b$. Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Let α be a real number. If $f(a) = f(b) = 0$, show that there exists $x_0 \in (a, b)$ such that $\alpha f(x_0) + f'(x_0) = 0$.

Part III

N.B. Each question in Part III carries 12 marks.

1. Let M_n be the $n \times n$ matrix with all 1's along the main diagonal, directly above the main diagonal and directly below the main diagonal and 0's everywhere else. For example,

$$M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \qquad \text{Let } d_n = \det M_n.$$

   (a) Find d_1, d_2, d_3, d_4.
   (b) Find a formula expressing d_n in terms of d_{n-1} and d_{n-2}, for all $n \geq 3$.
   (c) Find d_{100}.

2. Let $p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - 4x^{2n-3} + \cdots - 2nx + (2n+1)$. Show that the polynomial $p(x)$ has no real root.

3. Let $f(x) = x^{10} + a_1x^9 + a_2x^8 + \cdots + a_{10}$ where a_i's are integers. If all the roots of $f(x)$ are from the set $\{1, 2, 3\}$, determine the number of such polynomials. Further, if $g(x)$ is the sum of all such polynomials $f(x)$, then show that the constant term of $g(x)$ is $\frac{1}{2}(3^{12} + 1) - 2^{12}$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that
$$f(x+h) - f(x) = hf'(x + \frac{1}{2}h),$$
for all real x and h. Prove that f is a polynomial of degree atmost 2.

5. (a) Let $n = 9$. Express n as a sum of positive integers such that their product is maximum. Find the value of the maximum product.
(b) Repeat part (a) for $n = 10$ and $n = 11$.
(c) Given a positive integer $n \geq 6$, express n as a sum of positive integers such that their product is maximum. Find the value of the maximum product.