Instructions

1. The data analysis examination lasts for 4 hours and is worth a total of 150 marks.

2. Dedicated IOAA **Summary Answer Sheets** are provided for writing your answers. Enter the final answers into the appropriate boxes in the corresponding **Summary Answer Sheet**. On each Answer Sheet, please fill in
   - Student Code (Country Code and 1 digit)

3. **Graph Papers** are required for your solutions. On each Graph Paper, please fill in
   - Student Code (Country Code and 1 digit)
   - Question no.
   - Graph no. and total number of graph papers used.

4. There are **Answer Sheets** for carrying out detailed work/rough work. On each Answer Sheet, please fill in
   - Student Code (Country Code and 1 digit)
   - Question no.
   - Page no. and total number of pages.

5. Start each problem on a separate Answer Sheet. Please write only on the printed side of the sheet. Do not use the reverse side. If you have written something on any sheet which you do not want to be evaluated, cross it out.

6. Use as many mathematical expressions as you think that may help the evaluator to better understand your solutions. The evaluator may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).

7. You are not allowed to leave your working desk without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, need more Answer Sheets, etc.), please draw the attention of the invigilator using the sign card.

8. The beginning and end of the examination will be indicated by a long sound signal. Additionally, there will be a buzzer sound, fifteen minutes before the end of the examination (before the final sound signal).

9. At the end of the examination you must stop writing immediately. Sort and put your Summary Answer Sheets, Graph Papers, and Answer Sheets for each part (D1 and D2) in separate stack. You are not allowed to take any sheet of paper out of the examination area.

10. Wait at your table until your envelope is collected. Once all envelopes are collected, your student guide will escort you out of the examination area.

11. A list of constants is given on the next page.

Table of constants

Mass ($M_\oplus$)	5.98×10^{24} kg	**Earth**
Radius ($R_\oplus$)	6.38×10^{6} m	
Acceleration of gravity (g)	9.8 m s^{-2}	
Obliquity of Ecliptic	23°27’	
Length of Tropical Year	365.2422 mean solar days	
Length of Sidereal Year	365.2564 mean solar days	
Albedo	0.39	
Mass($M_☾$)	7.35×10^{22} kg	**Moon**
Radius ($R_☾$)	1.74×10^{6} m	
Mean distance from Earth	3.84×10^{8} m	
Orbital inclination with the Ecliptic	5.14°	
Albedo	0.14	
Apparent magnitude (mean full moon)	-12.74	
Mass ($M_\odot$)	1.99×10^{30} kg	**Sun**
Radius ($R_\odot$)	6.96×10^{8} m	
Luminosity ($L_\odot$)	3.83×10^{26} W	
Absolute Magnitude ($\mathcal{M}_\odot$)	4.80 mag	
Angular diameter	0.5 degrees	
1 au	1.50×10^{11} m	**Physical constants**
1 pc	206,265 au	
Distance from Sun to Barnard’s Star	1.83 pc	
Gravitational constant (G)	6.67×10^{-11} N · m^2 · kg^{-2}	
Planck constant (h)	6.62×10^{-34} J · s	
Boltzmann constant (k_B)	1.38×10^{-23} J · K^{-1}	
Stefan-Boltzmann constant (σ)	5.67×10^{-8} W · m^{-2} · K^{-4}	
Hubble constant (H_0)	72 km s^{-1} Mpc^{-1}	
Speed of light in vacuum (c)	299,792,458 m s^{-1}	
Proton mass	938.27 MeV · c^{-2}	
Deuterium mass	1875.60 MeV · c^{-2}	
Neutron mass	939.56 MeV · c^{-2}	
Helium-3 mass	2808.30 MeV · c^{-2}	
Helium-4 mass	3727.40 MeV · c^{-2}	

(D1) Calibrating the distance ladder to the LMC [75 marks]

An accurate trigonometric parallax calibration for Galactic Cepheids has long been sought, but is very difficult to achieve in practice. All known classical (Galactic) Cepheids are more than 250 pc away, therefore for direct distance estimates to achieve an uncertainty of up to 10%, parallax uncertainties of up to ±0.2 milliarcsec are needed, requiring space-based observations. The Hipparcos satellite reported parallaxes for 200 of the nearest Cepheids, but even the best of these had high uncertainties. Recent progress has come with the use of the Fine Guidance Sensor on HST where parallaxes (in many cases) accurate to better than ±10% were obtained for 10 Cepheids, spanning a range of periods from 3.7 to 35.6 days. These nearby Cepheids cover distances from about 300 to 560 pc.

The measured periods, P, and average magnitudes in V, K and I bands are given in **Table 1** as well as the A_V and A_K for extinction in V and K bands, respectively. The measured parallaxes with their uncertainties are also given in milliarcsec (mas). All measured apparent magnitudes have negligible uncertainty.

Table 1: Periods and average apparent magnitudes of 5 Galactic Cepheids with accurate parallax measurements.

	P (day)	**<V> (mag)**	**<K> (mag)**	**A_V (mag)**	**A_K (mag)**	**<I> (mag)**	**parallax (mas)**	**error (mas)**
RT Aur	3.728	5.464	3.925	0.20	0.02	4.778	2.40	0.19
FF Aql	4.471	5.372	3.465	0.64	0.08	4.510	2.81	0.18
X Sgr	7.013	4.556	2.557	0.58	0.07	3.661	3.00	0.18
ζ Gem	10.151	3.911	2.097	0.06	0.01	3.085	2.78	0.18
l Car	35.551	3.732	1.071	0.52	0.06	2.557	2.01	0.20

(D1.1) The observed correlation between the period of a Cepheid and its brightness is usually described by the so-called "Period-Luminosity (PL) relation", where $L \propto P^{\beta}$. In fact, such a relation is normally expressed in terms of the period and absolute magnitude, instead of luminosity. Hereafter, we shall refer to the Period-Absolute magnitude relation as the conventionally named "PL relation".

Use the data given in Table 1 to plot a suitable linear graph in order to derive the Cepheid PL relation for the V-band and K-band. You should plot each graph separately on different pieces of graph paper. Determine the slope of the line that best describes the linear relation of the data. (You may find the relation $\Delta(\log_{10} x) \approx \dfrac{\Delta x}{x \log_e 10}$ useful)

[36.5 Marks]

Any apparent differences in PL relations of stars in the different bands can be explained if one also considers differences in colour. Therefore, the PL relation is in fact a PLC (Period-Luminosity-Colour) relation. This is from the reddening effect, due to extinction being a function of wavelength, which can also vary among different Cepheids due to their different metallicities, foreground Interstellar Medium and dust.

A new reddening-free magnitude (or bandpass) called "Wesenheit" has been proposed that does not require the explicit information of the extinction of individual stars but uses colour information from the star itself to get rid of the effect. For example, W_{VI} use V and I band photometry and is defined as

$$W_{VI} = V - \left[\frac{A_V}{E(V-I)} \right] (V-I),$$

$$= V - R_v (V-I)$$

where R_v depends on the reddening law. In this case, we shall take the value of R_v to be 2.45.

(D1.2) From the data given in Table 1, plot and derive the reddening-free PL relation using Wesenheit W_{VI} magnitudes. Estimate the linear slope of the relation as well as its uncertainty. [14.5 Marks]

(D1.3) Next, we would like to use the newly-derived PL relations from question (D1.1) & (D1.2) to estimate the distance to the Large Magellenic Cloud (LMC) using periods and magnitudes of classical Cepheids in the LMC. In **Table 2**, the periods, average extinction-corrected apparent magnitudes**,** <V_{corr}>, and Wesenheit W_{VI} magnitudes are given.

Estimate the distance modulus, μ, to each star and then use all the information to derive the distance to the LMC (in parsecs) and its standard deviation for each band.

Compare if the derived distances are statistically different for the 2 bands (YES/NO).

Are the standard deviations of the estimated distances for 2 bands different (YES/NO)?

Based on this dataset, which band (V or Wesenheit) is more accurate in estimating the distance to the LMC? [24 Marks]

Table 2: Period, average extinction-corrected apparent magnitude**,** <V_{corr}>, and average Wesenheit magnitude measurements of Cepheids in the LMC

	P (day)	**<V_{corr}> mag**	**<W_{VI}> mag**
HV12199	2.63	16.08	14.56
HV12203	2.95	15.93	14.40
HV12816	9.10	14.30	12.80
HV899	30.90	13.07	10.97
HV2257	39.36	12.86	10.54

(D2) **The search for dark matter** **[75 marks]**

A low surface brightness galaxy (LSB) is a diffuse galaxy with a surface brightness that, when viewed from the Earth, is at least one magnitude lower than the ambient night sky.

Some of its matter is in the form of "baryonic" matter such as neutral hydrogen gas and stars. However, most of its matter is in the form of invisible mass – so called "dark matter". In this question, we will investigate the mass of dark matter in a galaxy, the effect of dark matter on the rotation curves of the galaxy, and the distribution of dark matter in the galaxy.

The table below provides the data of a LSB galaxy named UGC4325. The galaxy is assumed to be edge-on. At every distance r from the centre of the galaxy, we measure

1. λ_{obs}, the observed wavelength of the Hα emission line. The Hubble expansion of the Universe has already been excluded from the data.

2. V_{gas}, the contribution of the gas component to the rotation due to M_{gas}, derived from HI surface densities.

3. V_*, the contribution of the stellar component to the rotation due to M_*, derived from R-band photometry.

The rotational velocities of the test particle due to the gas component, V_{gas}, and the star component, V_*, are defined as the velocities in the plane of the galaxy that would result from the corresponding components without any external influences. These velocities are calculated from the observed baryonic mass density distributions.

r (kpc)	λ_{obs} (nm)	V_{gas} (km/s)	V_* (km/s)
0.70	656.371	2.87	20.97
1.40	656.431	6.75	32.22
2.09	656.464	14.14	40.91
2.79	656.475	20.18	46.75
3.49	656.478	24.08	50.10
4.89	656.484	28.08	47.94
6.25	656.481	29.25	45.47
7.10	656.481	27.03	47.78
9.03	656.482	25.90	45.32
12.05	656.482	21.03	42.30

The mass of dark matter $M_{\mathrm{DM}}(r)$ within a volume of radius r can be defined in terms of the rotational velocity due to dark matter V_{DM}, the radius r and gravitational constant G,

$$M_{\mathrm{DM}}(r) = \frac{rV_{\mathrm{DM}}^2}{G}. \tag{1}$$

To a good approximation, the observed rotational velocity V_{obs} can be modelled as

$$V_{\mathrm{obs}}^2 = V_{\mathrm{gas}}^2 + V_*^2 + V_{\mathrm{DM}}^2. \tag{2}$$

The observed rotational velocity V_{obs} depends on the mass of the galaxy $M(r)$ within a volume of radius r measured from the galaxy's centre.

The mass density $\rho_{\mathrm{DM}}(r)$ of dark matter within a volume of radius r can be modelled by a galaxy density profile,

$$\rho_{\mathrm{DM}}(r) = \frac{\rho_0}{1+\left(\frac{r}{r_{\mathrm{C}}}\right)^2} \tag{3}$$

where ρ_0 and r_{C} are the central density and the core radius of the galaxy, respectively. According to the density profile, the mass of dark matter $M_{\mathrm{DM}}(r)$ within a volume of a radius r can be described by

$$M_{\mathrm{DM}}(r) = 4\pi\rho_0 r_{\mathrm{C}}^2\left[r - r_C \arctan(r/r_C)\right]. \tag{4}$$

Part 1 The mass of dark matter and rotation curves of the galaxy

(D2.1) In laboratories on Earth, Hα has an emitted wavelength λ_{emit} of 656.281 nm. Compute the observed rotational velocities of the galaxy V_{obs} and the rotational velocities due to the dark matter V_{DM} at distance r in units of km s^{-1}.

For the different values of r given in the table, compute the dynamical mass $M(r)$ and the mass of dark matter $M_{\mathrm{DM}}(r)$ in solar masses. [21]

(D2.2) Create rotation curves of the galaxy on graph paper by plotting the points of V_{obs}, V_{DM}, V_{gas}, V_* versus the radius r and draw smooth curves through the points (mark your graph as "D2.2").

Order the contribution of the different components to the observed velocity in descending order. [16]

Part 2 Dark matter distribution

(D2.3) Take a data point at small r and large r to estimate ρ_0 and r_C. Note that for large values of x, $\arctan(x) \approx \pi / 2$ and at small x, $\arctan(x) \approx x - x^3 / 3$. [7]

(D2.4) By comparing Equation (4) to a linear function, the central density ρ_0 could also be found by a linear fit. Plot an appropriate graph so that a linear fit can be used to find another value of ρ_0. Evaluate ρ_0 in units of M⊙ kpc^{-3}. (Mark your graph as "D2.4"). If you cannot find the value of r_C from the previous part, use r_C = 3.2 kpc as an estimate for this part. [19]

(D2.5) Compute logarithmic values of the dark matter density, $\ln[\rho_{DM}(r)]$, and plot the distribution of dark matter in the galaxy as a function of radius r on graph paper. (Mark your graph as "D2.5"). [12]