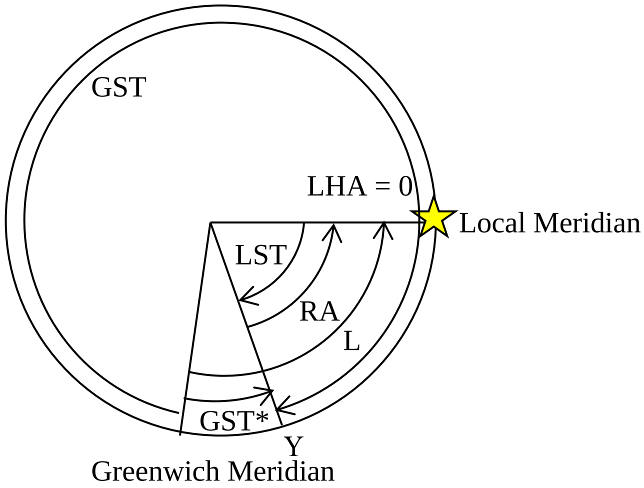
(T1) The Large Magellanic Cloud in Phuket **[10 marks]**

The coordinates of the Large Magellanic Cloud (LMC) are R.A. = 5h 24min and Dec = $-70^{\circ}00'$. The latitude and longitude of Phuket are $7^{\circ}53'$ N and $98^{\circ}24'$ E, respectively. What is the date when the LMC culminates at 9pm as seen from Phuket in the same year? You may note that the Greenwich sidereal time, GST, at 00h UT 1st January is about 6h 43min, and Phuket follows UT+7 time zone.

Solution:



When it culminates, $\text{LHA} = 0, \quad \text{RA} = -\text{LST}$ [1.0]

$$\text{LST} = -(\text{L} - \text{GST}^*) = -\text{RA}$$

$$\text{GST}^* = \text{L} - \text{RA} = \text{6h 34min} - \text{5h 24min} = \text{1h 10min}$$ [2.0]

$$\text{GST} = \text{24h} - \text{1h 10min}$$

$$\text{GST} = \text{22h 50min}$$ [1.0]

Phuket time of 21:00 is UT14:00. [1.0]

$$\text{GST} = \text{GST of 1}^{\text{st}}\text{ Jan} + \text{The angle that } \Upsilon \text{ moves away from 1}^{\text{st}}\text{ Jan} + \text{UT}$$ [1.0]

$$\text{22h 50min} = \text{6h 43min} + \left[\left(\frac{\text{Days that away from 1}^{\text{st}}\text{Jan}}{\text{365.2422 days}}\right) \times \text{24 h}\right] + \left[\text{14h} \times \left(\frac{24}{23.9344}\right)\right]$$

$$\text{Days that away from 1}^{\text{st}}\text{Jan} = (\text{22h 50min} - \text{6h 43min} - \text{14h 2min})\left(\frac{\text{365.2422 days}}{\text{24h}}\right)$$

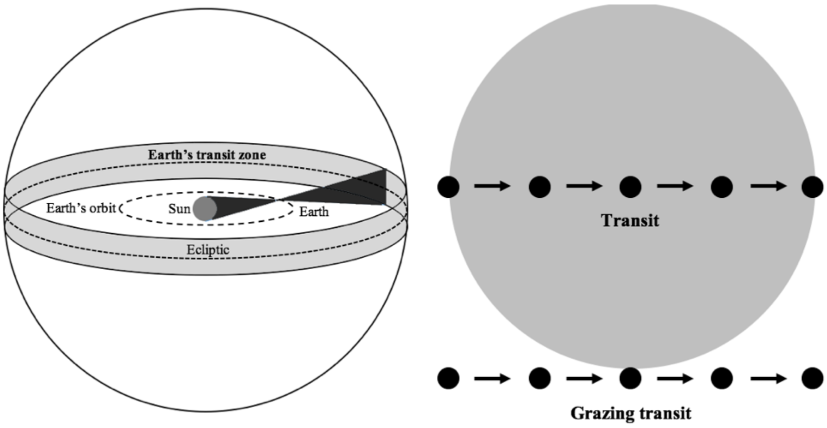
[2.0]

$$\text{Days that away from 1}^{\text{st}}\text{Jan} = \text{31.7 days from 1}^{\text{st}}\text{Jan (at Greenwich)}$$ [1.0]

The time of the year that the LMC culminates in Phuket is on 2nd February. [1.0]

(T2) Earth's transit zone **[10 marks]**

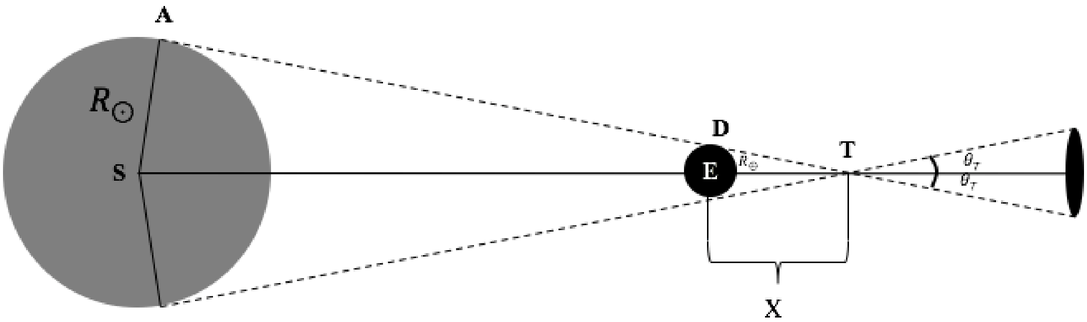
Earth's transit zone is an area where extrasolar observers (located far away from the Solar system) can detect the Earth transiting across the Sun. For observers on the Earth, this area is the projection of a band around the Earth's ecliptic onto the celestial plane (light grey area in the left figure). Assume that the Earth has a circular orbit of 1 au.



a) Find the angular width of that part of the Earth's transit zone in degrees, where the extrasolar observers can detect Earth's total transit (the whole Earth's disk passing in front of the Sun). [5]

Solution:

For Earth's transit, the whole Earth's disk should pass in front of the Sun



From $\Delta STA \sim \Delta ETD$: **[1.0]**

$$\frac{X}{R_\oplus} = \frac{a+X}{R_\odot}$$

$$X = \frac{aR_\oplus}{R_\odot - R_\oplus}$$ **[1.0]**

The half angular size of the Earth's Transit Zone with transit can be written as,

$$\theta_T = \arcsin\left(\frac{R_\oplus}{X}\right) = \arcsin\left(\frac{R_\odot}{a+X}\right) = \arcsin\left(\frac{R_\odot - R_\oplus}{a}\right)$$ **[1.0]**

Solution with $\theta_T = \arctan\left(\frac{R_\odot - R_\oplus}{a}\right) \approx \frac{R_\odot - R_\oplus}{a}$ ***is acceptable with full mark***

The angular size of the Earth's Transit Zone with transit is

$$2\theta_T = 2\arcsin\left(\frac{R_\odot - R_\oplus}{a}\right)$$ **[1.0]**

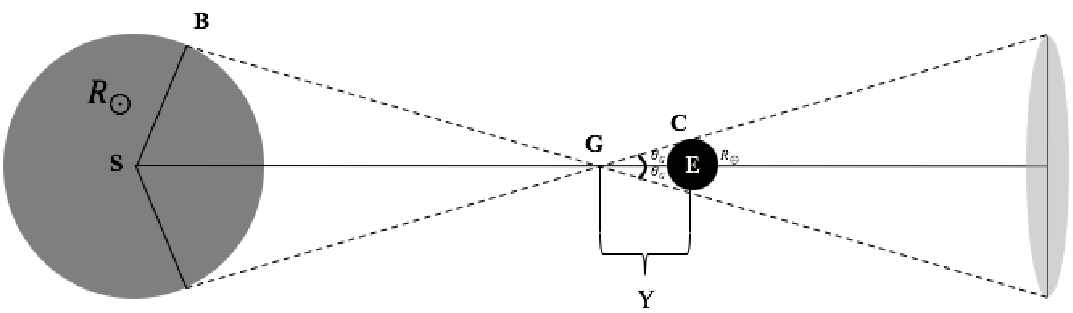
$$2\theta_T = 0.527^\circ$$ **[1.0]**

b) Find the angular width of that part of Earth's transit zone in degrees, where the extrasolar observers can detect at least Earth's grazing transit (any part of the Earth's disk passing in front of the Sun).

[5]

Solution:

For Earth's grazing transit, any part of the Earth's disk should pass in front of the Sun



From $\Delta SGB \sim \Delta EGC$: **[1.0]**

$$\frac{Y}{R_\oplus} = \frac{a - Y}{R_\odot}$$

$$Y = \frac{aR_\oplus}{R_\odot + R_\oplus}$$ **[1.0]**

The angular size of the Earth's Transit Zone with transit can be written as,

$$\theta_G = \arcsin\left(\frac{R_\oplus}{Y}\right) = \arcsin\left(\frac{R_\odot}{a - Y}\right) = \arcsin\left(\frac{R_\odot + R_\oplus}{a}\right)$$ **[1.0]**

Solution with $\theta_G = \arctan\left(\frac{R_\odot + R_\oplus}{a}\right) \approx \frac{R_\odot + R_\oplus}{a}$ ***is acceptable with full mark***

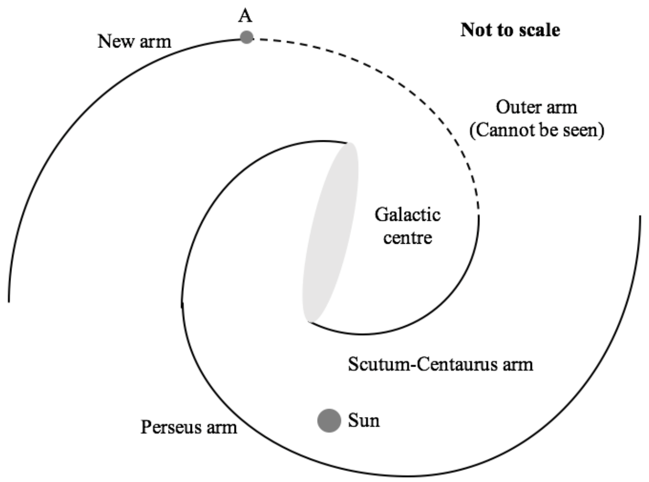
The angular size of the Earth's Transit Zone with grazing transit is

$$2\theta_G = 2\arcsin\left(\frac{R_\odot + R_\oplus}{a}\right)$$ **[1.0]**

$$2\theta_G = 0.537^\circ$$ **[1.0]**

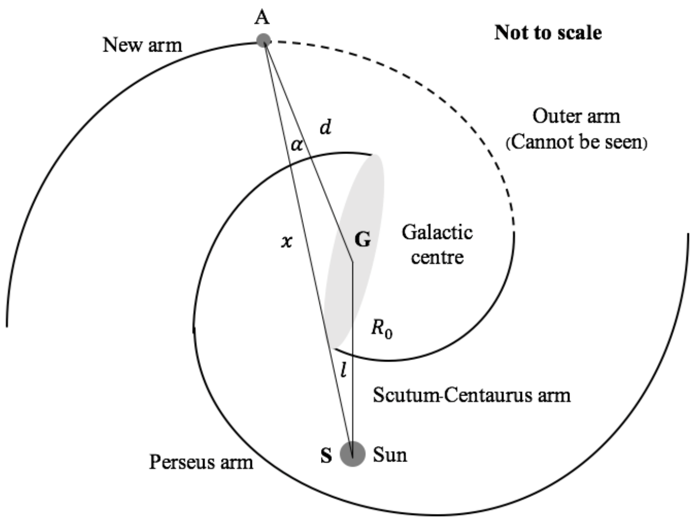
(T3) Milky Way New Far Outer Arm **[10 marks]**

In 2011, Dame and Thaddeus found a new part of outer arm of the Milky Way by studying the CO line using the CfA 1.2m telescope. They detected that the distribution of CO starts at galactic longitude $\ell = 13.25^{\circ}$ (marked **A** in the figure) where it has radial velocity of 20.9 km s^{-1} towards the Sun. Assume that the galactic rotation curve is flat beyond 5 kpc from the galactic centre. The distance between the Sun and the Galactic centre is 8.5 kpc. The velocity of the Sun around the Galactic centre is 220 km s^{-1}.



a) Find the distance from the start of the arm (point **A**) to the Galactic centre. [7]

Solution:



If there is the flat rotation curve beyond 2 kpc from the Galactic centre, the orbital velocity of the Sun and the start of the arm are both equal to

$$v_r = -v_{\text{LSR}} \sin(\ell) + v_{\text{LSR}} \sin(\alpha)$$ **[3.0]**

Sine law (ΔAGS): $\dfrac{\sin(\ell)}{d} = \dfrac{\sin(\alpha)}{R_0}$ **[1.0]**

$v_r = -v_{\text{LSR}} \sin(\ell) + v_{\text{LSR}} R_0 \dfrac{\sin(\ell)}{d}$ **[1.0]**

$20.9 = 220 \sin(13.25°) - 220 \times 8.5 \times \dfrac{\sin(13.25°)}{d}$ **[1.0]**

Correctly substitute numerical values

$d = 14.5\,\text{kpc}$ **[1.0]**

b) Find the distance from the start of the arm (point **A**) to the Sun. [3]

Solution:

Cosine law (ΔAGS):

$d^2 = x^2 + R_0^2 - 2xR_0 \cos(\ell)$ **[1.0]**

$14.5^2 = x^2 + 8.5^2 - 2 \times 8.5 x \cos(13.25^\circ)$ **[1.0]**

Correctly substitute numerical values

$$x = \frac{17\cos(13.25^\circ) + \sqrt{17^2\cos^2(13.25^\circ) - 4 \times (8.5^2 - 14.5^2)}}{2}$$

$x = 22.6\,\text{kpc}$ **[1.0]**

(T4) 21-cm HI galaxy survey **[10 marks]**

A radio telescope is equipped with a receiver which can observe in a frequency range from 1.32 to 1.52 GHz. Its detection limit is 0.5 mJy per beam for a 1-minute integration time. In a galaxy survey, the luminosity of the HI spectral line of a typical target galaxy is 10^{28} W with a linewidth of 1 MHz. For a large beam, the HI emitting region from a far-away galaxy can be approximated as a point source. The HI spin-flip spectral line has a rest-frame frequency of 1.42 GHz.

What is the highest redshift z of a typical HI galaxy that can be detected by a survey carried out with this radio telescope, using a 1-minute integration time? You may assume in your calculation that the redshift is small and the non-relativistic approximation can be used. Note that $1\ \text{Jy} = 10^{-26}\ \text{Wm}^{-2}\text{Hz}^{-1}$.

Solution:

Case 1: If we only consider the frequency range where HI spectral line can be detected by the receiver of this radio telescope,

$$z \approx \frac{f_0 - f}{f_0}$$ [**1 Mark**]

The lowest frequency, 1.32 GHz, can be used to observe the highest redshifted HI, at

$$z = \frac{1.42 - 1.32}{1.42} = 0.0704$$ [**1 Mark**]

Alternative solution: Above is the cosmological redshift formula in frequency conventionally used in radio astronomy. However, if the cosmological redshift in optical astronomy is used, student may use

$$z \approx \frac{f_0 - f}{f} = \frac{1.42 - 1.32}{1.32} = 0.0758$$

and the student should be awarded full mark for this part.

Case 2: Use the information given to calculate the redshift limit set by the flux limit due to furthest distance that typical HI galaxy can be detected by the telescope

for low z, non-relativistic redshift $d = \frac{v_r}{H_0} = \frac{cz}{H_0}$ [**1 Mark**]

The HI spectral-line flux density for a galaxy with HI luminosity L and line-width Δf at distance d is

$$\frac{L}{4\pi d^2} \times \frac{1}{\Delta f} (\text{Wm}^{-2}\text{Hz}^{-1})$$ [**2 Mark**]

Setting this to the detection limit and solve correctly, and taking special care of converting various units

$$S = \frac{LH_0^2}{4\pi\Delta f c^2 z^2} \geq S_{\mathrm{lim}} = 0.5\times10^{-26}$$ [**1 Mark**]

Therefore,

$$z \leq \frac{H_0}{c}\sqrt{\frac{L}{4\pi\Delta f S_{\mathrm{lim}}}},$$ [**1 Mark**]

Correctly substitute numerical values,

$$z \leq \frac{67.8(\mathrm{km^{-1}Mpc^{-1}})}{2.99\times10^5(\mathrm{km\ s^{-1}})\times3.08\times10^{22}\ (\mathrm{m\ Mpc^{-1}})}\sqrt{\frac{10^{28}\ \mathrm{W}}{4\pi\times10^6(\mathrm{Hz})\times0.5\times10^{-29}\ (\mathrm{Wm^{-2}Hz^{-1}})}}$$

[**1 Mark**]

$$z \leq 0.0929$$ [**1 Mark**]

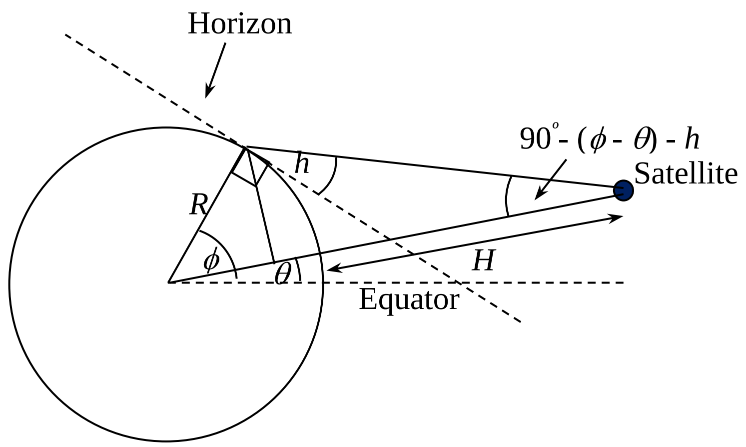
Conclusion: For typical HI galaxy, the highest redshift is limited by the receiver's lowest frequency limit and we get z_{max}=0.0704 (or 0.0758 for alternative solution given above)
[**1 Marks**]

The last part is marked only if student calculate both instrumental factors.
If students use more precise cosmological effects, they will be marked accordingly.

(T5) A Synchronous Satellite **[10 marks]**

A synchronous satellite is a satellite which orbits the Earth with its period exactly equal to the period of rotation of the Earth. The height of these satellites is 35,786 km above the surface of the Earth. A satellite is put at an inclined synchronous orbit with the inclination $\theta = 6.69°$ to the equatorial plane. Calculate the precise value of the possible maximum altitude of the satellite for an observer at latitude of $\phi = 51.49°$.

Solution:



Draw diagram and correctly identify the altitude [1.0]

Identify another angle with the altitude *h* in its expression, such as $90° - (\phi - \theta) - h$ [1.0]

Maximum altitude occurs when the satellite is at meridian [1.0]

Intermediate steps that can lead to the calculation of *h*, such as

$$\tan(90° - (\phi - \theta) - h) = \frac{R\sin(\phi - \theta)}{H + R - R\cos(\phi - \theta)}$$ [3.0]

Correct expression for *h*: [2.0]

$$h = 90° - (\phi - \theta) - \tan^{-1}\left(\frac{R\sin(\phi - \theta)}{H + R - R\cos(\phi - \theta)}\right)$$

Correct value of *h*

$$h = 90° - (51.49^{\circ} - 6.69^{\circ}) - \tan^{-1}\left(\frac{\sin(51.49^{\circ} - 6.69^{\circ})}{\frac{35,786 \text{ km}}{6.38\times 10^{3} \text{ km}} + 1 - \cos(51.49^{\circ} - 6.69^{\circ})}\right)$$

$h = 38.4°$ [2.0]

Maximum of 2.0 marks for solution with $h = 90.0° - 51.49° + 6.69° = 45.2°$

(T6) Supernova 1987A **[15 marks]**

Supernova SN 1987A was at its brightest with apparent magnitude of +3 on about 15thMay 1987 and then faded, finally becoming invisible to the naked eye by 4th February 1988. It is assumed that brightness B varied with time t as an exponential decline, $B = B_0 e^{-t/\tau}$, where B_0 and τ are constant. The maximum apparent magnitude which can be seen by the naked eye is +6.

a) Determine the value of τ in days. [5]

Solution:

$$m - m_0 = -2.5\log_{10}\left(\frac{B}{B_0}\right)$$ **[1.0]**

$$m - m_0 = 2.5\frac{t}{\tau}\log_{10} e$$ **[1.0]**

The days from 15thMay 1987 to 4th February 1988 is 265 days **[1.0]**

$$6 - 3 = 2.5\frac{265\text{ days}}{\tau}\log_{10} e$$ **[1.0]**

$$\tau = 95.9\text{ days}$$ **[1.0]**

b) Find the last day that observers could have seen the supernova if they had a 6 inch (15.24 cm) telescope with transmission efficiency $T = 70\%$. Assume that the average diameter of the human pupil is 0.6 cm. [10]

Solution:

Consider the energy transmission

$$B_e d^2 = T B_T D^2$$ **[3.0]**

$$\frac{B_e}{B_T} = T\frac{D^2}{d^2}$$

$$m - m_0 = -2.5\log_{10}\left(\frac{B}{B_0}\right)$$ **[1.0]**

$$m_{\lim} = m_e + 2.5\log_{10} T + 5\log_{10}\left(\frac{D}{d}\right)$$ **[1.0]**

$$m_{\lim} = 6 + 2.5\log_{10} 0.7 + 5\log_{10}\left(\frac{15.24\text{ cm}}{0.6\text{ cm}}\right)$$

$$m_{\lim} = 12.64$$

and

From $m - m_0 = 2.5\frac{t}{\tau}\log_{10} e$ **[2.0]**

$12.64 - 3 = 2.5\frac{t}{95.9\text{ days}}\log_{10} e$ **[1.0]**

$t = 851.5\text{ days}$ **[1.0]**

The last day that observers could have seen the supernova is on **12th September 1989**

[1.0]

11th - 13th September 1989 are acceptable

(T7) Life on Other Planets **[20 marks]**

One place to search for life is on planets orbiting main sequence stars. A good starting point is the planets that have an Earth-like temperature range and a small temperature fluctuation. Assume that for a main sequence star, the relation between the luminosity L and the mass M is given by

$$L \propto M^{3.5}.$$

You may assume that the total energy E released over the lifetime of the star is proportional to the mass M of the star. For the Sun, it will have a main sequence lifetime of about 10 billion years. The stellar spectral types are given in the table below. Assume that the spectral subclasses of stars (0-9) are assigned on a scale that is linear in $\log M$.

Spectral Class	O5V	B0V	A0V	F0V	G0V	K0V	M0V
Mass ($M_\odot$)	60	17.5	2.9	1.6	1.05	0.79	0.51

a) If it takes at least 4×10^9 years for an intelligent life form to evolve, what is the spectral type (accurate to the subclass level) of the most massive star in the main sequence around which astronomers should look for intelligent life? [6]

Solution:

Since $E \propto M \rightarrow L\tau \propto M$ and $L \propto M^{3.5}$,

$$\tau = \tau_\odot \left(\frac{M_\odot}{M}\right)^{2.5}$$ **[2.0]**

The maximum mass of the star should be

$$M = M_\odot \left(\frac{\tau_\odot}{\tau}\right)^{1/2.5} = M_\odot \left(\frac{10^{10}}{4\times10^9}\right)^{1/2.5}$$ **[0.5]**

$$M = 1.44 M_\odot$$ **[0.5]**

Use of spectral class-log M linear relation

$$\frac{10}{(\log 1.6 - \log 1.05)} \times (\log 1.6 - \log 1.44) = 2.5$$ **[2.0]**

which corresponds to the star of type **F2V** to **F3V** **[1.0]**

0.5 mark for the answer "between F0V and G0V"

b) Assume that the target planet has the same emissivity ε and albedo a as the Earth. In order to have the same temperature as the Earth, express the distance d, in au, of the planet to its parent main sequence star, of mass M. [6]

Solution:

The received power by a planet of a radius r and a distance d from the star is

$$P_{input} = (1-a)\frac{\pi r^2 L}{4\pi d^2} = (1-a)\frac{r^2 L}{4d^2}$$, where a is the planet's albedo.**[1.0]**

The released power of blackbody radiation is

$$P_{output} = 4\pi r^2 \varepsilon\sigma T^4$$, where ε is emissivity. **[1.0]**

Since we assume that the planet emits heat as blackbody radiation,

$$(1-a)\frac{r^2 L}{4d^2} = 4\pi r^2 \varepsilon\sigma T^4 .$$ **[1.0]**

The temperatures of the planet and the Earth are similar,

$$T^4 = \frac{(1-a)L}{16\pi\varepsilon\sigma d^2} = \frac{(1-a)L_\odot}{16\pi\varepsilon\sigma d_\odot^{\,2}} .$$

$$\frac{L}{d^2} = \frac{L_\odot}{d_\odot^{\,2}}$$ **[1.0]**

$$\left(\frac{d}{d_\odot}\right)^2 = \frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^{3.5}$$ **[1.0]**

The distance of the planet to the star is $d = \left(\frac{M}{M_\odot}\right)^{3.5/2}$ au. **[1.0]**

c) The existence of a planet around a star can be shown by the variation in the radial velocity of the star about the star-planet system centre of mass. If the smallest Doppler shift in the wavelength detectable by the observer is $(\Delta\lambda/\lambda) = 10^{-10}$, calculate the lowest mass of such a planet in b), in units of Earth masses, that can be detected by this method, around the main sequence star in a). [8]

Solution:

Angular speed around the center of mass can be derived by Kepler's law.

$$T^2 = \frac{4\pi^2 d^3}{G(M+m)}$$ **[1.0]**

$$\omega = \sqrt{\frac{G(M+m)}{d^3}}$$ **[1.0]**

According to the centre of mass of the planet and the star,

$$Md_s = md$$

$$d_s = \frac{m}{M} d$$ **[1.0]**

and

$$v = \omega d_s$$ **[1.0]**

$$v = \omega d \frac{m}{M} = \frac{m}{M}\sqrt{\frac{G(M+m)}{d}} \approx m\sqrt{\frac{G}{Md}}$$

From $\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{m}{c}\sqrt{\frac{G}{Md}}$ **[1.0]**

Use of previously derived expression The distance of the planet to the star is

$$d = \left(\frac{M}{M_\odot}\right)^{3.5/2} \text{ au.}$$ **[1.0]**

$$m = \frac{\Delta\lambda}{\lambda} c\sqrt{\frac{Md}{G}} = \frac{\Delta\lambda}{\lambda} c\sqrt{\frac{M\left(\frac{M}{M_\odot}\right)^{3.5/2} \times 1 \text{ au}}{G}}$$

$$= \frac{\Delta\lambda}{\lambda} c\sqrt{\frac{M\,(1.44)^{3.5/2} \times 1 \text{ au}}{G}}$$

$$= (299792458\text{m/s}) \times 10^{-10}\sqrt{\frac{1.44 \times 1.99 \times 10^{30}\text{ kg }(1.44)^{3.5/2} \times 1.50 \times 10^{11}\text{m}}{6.67 \times 10^{-11}\text{ m}^3\text{kg}^{-1}\text{s}^{-2}}}$$ **[1.0]**

$$m = 3.31 \times 10^{24}\text{kg} = 0.554 M_\oplus$$ **[1.0]**

(T8) Star of Bethlehem **[20 marks]**

A great conjunction is a conjunction of Jupiter and Saturn for observers on Earth. Assume that Jupiter and Saturn have circular orbits in the ecliptic plane.

The time between successive conjunctions may vary slightly as viewed from the Earth. However, the average time period of the great conjunctions is the same as that of an observer at the centre of the Solar system.

a. Find the average great conjunction period (in years) and average heliocentric angle between two successive great conjunctions (in degrees). [6]

Solution:

Assume that the average Earth's orbit is at the Sun. Therefore, average great conjunction period equals to the synodic period between Jupiter and Saturn.

$$\frac{1}{P_{Synodic}} = \frac{1}{P_{Jupiter}} - \frac{1}{P_{Saturn}}$$ **[3.0]**

$$\frac{1}{P_{Synodic}} = \frac{1}{11.86} - \frac{1}{29.45}$$

$P_{Synodic} = 19.86$ years **[1.0]**

From average great conjunction period, in 19.86 years Jupiter and Saturn move

$$\text{Heliocentric angle} = \frac{19.86}{11.86} \times 360^{\circ} = 602.7^{\circ} = 242.7^{\circ}$$

in an eastward direction through the zodiac, or $360^{\circ} - 242.7^{\circ} = 117.3^{\circ}$ in a westward direction through the zodiac. **[2.0]**

b. The next great conjunction will be on 21st December 2020 with an elongation of 30.3° East of the Sun. Estimate in which constellation will the conjunction on 21st December 2020 occur? (Give the IAU Latin name or IAU three-letter abbreviation of the constellation, i.e. Ursa Major or UMa) [2]

Solution:

On 21st December 2020 (winter solstice), the Sun will be in the constellation of Sagittarius. Planets are always near the ecliptic. The current 12 zodiac constellations can be roughly divided into 12 equal parts of 30 degrees each.

The conjunction will locate elongation 30.3° East to the Sun. Therefore, the conjunction will be in the constellation of **Capricornus (Cap)**. **[2.0]**

Justification is NOT necessary for the full mark for this part

In 1606, Johannes Kepler determined that in some years the great conjunction can be happen thrice in the year due to the retrograde motions of the planets. He also determined that such an event happened in the year 7BC, which could have been the event commonly known as “The Star of Bethlehem”. For the calculations below you may ignore the precession of the axis of the Earth.

c. Estimate in which constellation did the great conjunctions in 7 BC occur? (Give the IAU Latin name or IAU three-letter abbreviation of the constellation, i.e. Ursa Major or UMa) [8]

Solution:

The duration between 7BC and 2020 is 2026 years or 2027 years .

Each conjunction occurs on the average every 19.86 years. Therefore, the number of conjunction between 7BC and 2020 is

Integer value of $2026/19.86 = 102$ **[2.0]**

Correct number of conjunctions 1.0 mark.
Correct number of conjunctions with integer 2.0 marks.

From a conjunction to its previous successive conjunction, the conjunction moves $243°$ westwards. The position of the conjunction in 7BC is

$102 \times 242.7° = 24758.33° = 278.3°$ **[3.0]**

westward, or equivalently $81.7°$ eastward from the position of the conjunction in 2020. Therefore, the conjunction was in the constellation of **Aries (Ari)** or **Pisces (Psc)** **[3.0]**

Justification is NOT necessary for the full mark for this part
Correct answer only

d. At the second conjunction of the series of three conjunctions in 7 BC, for the observer on Earth, estimate in which constellation was the Sun? (Give the IAU Latin name or IAU three-letter abbreviation of the constellation, i.e. Ursa Major or UMa) [4]

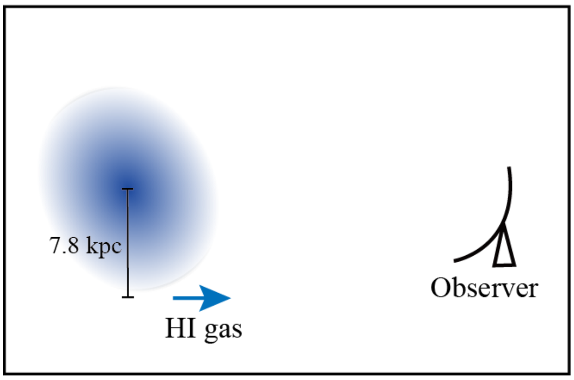
Solution:

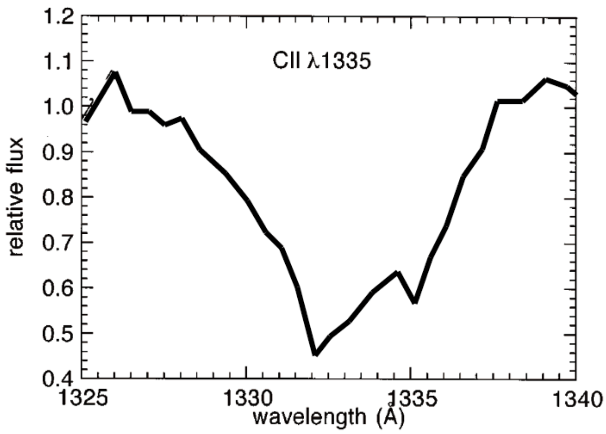
At the second conjunction, Jupiter and Saturn had retrograde motions. For observers on the Earth, both of them were in opposition. Therefore, the sun was in the constellation of **Libra (Lib) (if answer c) is Aries)** or **Virgo (Vir) (if answer c) is Pisces).** **[4.0]**
This question is marked only if students get c) correctly

(T2) Galactic Outflow **[20 marks]**

Cannon et al. (2004) conducted an HI observation of a disk starburst galaxy, IRAS 0833+6517, with the Very Large Array (VLA). The galaxy is located at a distance of 80.2 Mpc with an approximate inclination angle of 23 degrees. According to the HI velocity map, IRAS 0833+6517 appears to be undergoing regular rotation with the observed radial velocity of the HI gas of roughly 5850 km s^{-1} at a distance of 7.8 kpc from the centre (the left panel of the figure below).

Gas outflow from IRAS 0833+6517 is traced by using the blueshifted interstellar absorption lines observed against the backlight of the stellar continuum (the right panel of the figure). Assuming that this galaxy is gravitationally stable and all the stars are moving in circular orbits,





a) Determine the rotational velocity (v_{rot}) of IRAS 0833+6517 at the observed radius of HI gas. [5]

Solution:

The systemic or recessional velocity of IRAS 0833+6517 can be determined by the Hubble's law

$$v_{sys} = H_0 d$$

$$= 67.8 \text{ km s}^{-1}\text{Mpc}^{-1} \times 80.2 \text{ Mpc} \quad \textbf{[2]}$$

$$= 5437.6 \text{ km s}^{-1}$$

The observed radial velocity can be written in terms of systemic velocity and rotational velocity of a galaxy as

$$v_{rad} = v_{sys} + v_{rot} \sin i, \quad \textbf{[1]}$$

where v_{rad}, v_{sys}, and v_{rot} are the radial, systemic, and rotational velocity of the galaxy and i is an inclination angle.

The rotational velocity of the galaxy can thus be obtained by

$$v_{\text{rot}} = \frac{v_{\text{rad}} - v_{\text{sys}}}{\sin i} = \frac{5850 - 5437.6 \text{ km s}^{-1}}{0.39}$$ **[2]**

$$= 1055 \text{ km s}^{-1}$$

Answers between 1050 to 1060 are acceptable for full marks.

b) Calculate the escape velocity for a test particle in the gas outflow at the radius of 7.8 kpc. [9]

Solution:

Method 1:

In order to determine the escape velocity, we need to know the dynamical mass of the system. In case of the rotation dominated galaxy, a dynamical mass can be estimated as

$$M_{\text{dyn}}(<R) = v_{\text{rot}}^2 R / G,$$ **[2]**

where $M_{\text{dyn}}(<R)$ is the dynamical mass with in the radius R.

So the dynamical mass of IRAS 0833+6517 within the radius of 7.8 kpc is

$$M_{\text{dyn}}(<R) = \frac{(1055\times10^3 \text{m s}^{-1})^2 \times 7.8\times10^3 \text{pc} \times 206265 \text{au/pc} \times 1.50\times10^{11} \text{m/au}}{6.67\times10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}}$$ **[2]**

$$= 4.0\times10^{42} \text{kg}$$

or $M_{\text{dyn}}(<R) = 2.0\times10^{12} \text{M}_{\odot}$.

Assuming the virial theorem, the escape velocity can be written as

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$ **[2]**

Therefore, the escape velocity is

$$v_{\text{esc}} = \sqrt{\frac{2\times6.67\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2} \times 4.0\times10^{42}\text{kg}}{7.8\times10^3 \text{pc} \times 206265 \text{ au/pc} \times 1.50\times10^{11} \text{m/au}}}$$ **[3]**

$$= 1492 \text{ km s}^{-1}$$

Method 2:

In order to determine the escape velocity, we need to know the dynamical mass of the system. In case of the rotation dominated galaxy, a dynamical mass can be estimated as

$$M_{\text{dyn}}(<R) = v_{\text{rot}}^2 R / G,$$ **[2]**

where $M_{\rm dyn}(<R)$ is the dynamical mass with in the radius R. According to the virial theorem, the escape velocity is

$$v_{\rm esc} = \sqrt{\frac{2GM}{R}}$$ **[2]**

Substituting the dynamical mass into the above equation, we get

$$v_{\rm esc} = \sqrt{\frac{2GM}{R}} = \sqrt{2v_{\rm rot}^2}$$ **[3]**

$$= \sqrt{2\times(1055\ {\rm km\ s^{-1}})^2}$$
$$= 1492\ {\rm km\ s^{-1}}$$

[2]

Answers between 1487 to 1497 are acceptable for full marks.

c) Examine if the outflowing gas can escape from the galaxy at this radius by considering the velocity offset of the C II λ1335 absorption line, which is already corrected for the recessional velocity. (The central wavelength of the CII absorption line in labs is 1335 Å.) [6]

Solution:

To examine whether or not outflowing gas can escape from the galaxy, one needs to derive the outflow velocity of the gas from the blueshifted absorption line.

The outflow velocity or the velocity offset of any emission or absorption lines can be calculated by the following equation:

$$\frac{\Delta v}{c} = \frac{\Delta\lambda}{\lambda}$$ **[2]**

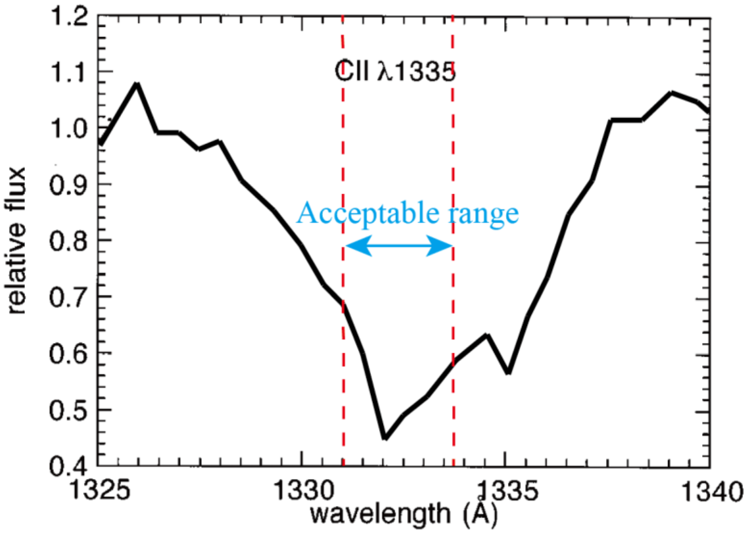


Figure 1 The CII absorption line with the acceptable range of the blueshifted wavelengths.

For the C II $\lambda 1335$ absorption line (Figure 1), we obtain

$$\Delta v_{\text{obs}}(\text{C II}\lambda 1335) = 299792.458 \text{ km s}^{-1}\left(\frac{1332-1335}{1335}\right) \qquad \textbf{[2]}$$

$$= -673.7 \text{ km s}^{-1}$$

Answers between -300 km s^{-1} ***to*** -900 km s^{-1} ***are acceptable with full marks.***

The velocity offsets of the C II $\lambda 1335$ absorption line is -673.7 km s^{-1}. It means that the outflowing gas is going out of the galaxy at this speed, which is smaller than the escape velocity obtained in (b).

We conclude that the outflowing gas CANNOT escape from IRAS 0833+6517. [2]

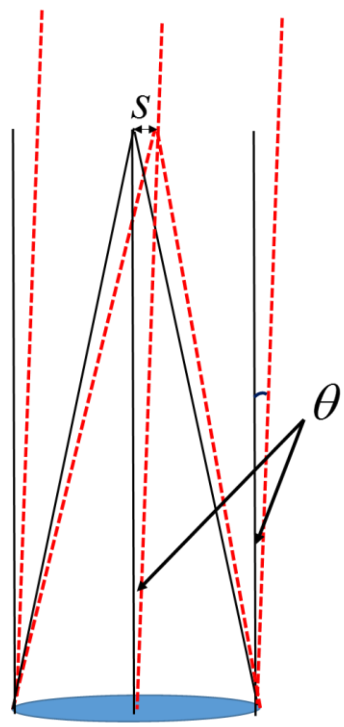
(T10) GOTO **[25 marks]**

The Gravitational-Wave Optical Transient Observer (GOTO) aims to carry out searches of optical counterparts of any Gravitational Wave (GW) sources within an hour of their detection by the LIGO and VIRGO experiments. The survey needs to cover a big area on the sky in a short time to search all possible regions constrained by the GW experiments before the optical burst signal, if any, fades away. The GOTO telescope array is composed of 4 identical reflective telescopes, each with 40-cm diameter aperture and f-ratio of 2.5, working together to image large regions of the sky. For simplicity, we assume that the telescopes' field-of-view (FoV) do not overlap with one another.

a) Calculate the projected angular size per mm at the focal plane, i.e. plate scale, of each telescope. [5]

Solution

For any two points on the sky separated by small angular distance θ, at the focal plane of the telescope the distance is S



$$\tan\theta = \frac{S}{f},$$ [**0.5 Mark**]

where f is the focal length

for small angle θ

$$\tan\theta \approx \theta$$

Then plate scale is

$$\frac{\theta}{S} = \frac{1}{f}$$ [**0.5 Mark**]

$$f = \text{F-ratio} \times \text{Diameter}$$ **[1 Mark]**

$$\text{Plate scale} = \frac{1}{\text{F-ratio} \times \text{Diameter}}$$ **[1 Mark]**

If student correctly quote the formula for plate scale will also get the full mark for the above calculation

$$\text{Plate Scale} = \frac{1}{2.5\times 40\times 10^{-2}(\text{m})\times 10^{3}(\text{mm m}^{-1})}\times\frac{180\times 3600\ (\text{arcsec})}{\pi}$$ [1 Mark]

= 206 arcsec/mm

Plate Scale = 3.44 arcmin/mm **[1 Mark]**

b) If the zero-point magnitude (i.e. the magnitude at which the count rate detected by the detector is 1 count per second) of the telescope system is 18.5 mag, calculate the minimum time needed to reach 21 mag at Signal-to-Noise Ratio (SNR) = 5 for a point source. We first assume that the noise is dominated by both the Read-Out Noise (RON) at 10 counts/pixel and the CCD dark (thermal) noise (DN) rate of 1 count/pix/minute. The CCDs used with the GOTO have a 6-micron pixel size and gain (conversion factor between photo-electron and data count) of 1. The typical seeing at the observatory site is around 1.0 arcsec. [8]

The signal to Noise ratio is defined by

$$\text{SNR} \equiv \frac{\text{Total Source Count}}{\sqrt{\Sigma_i \text{Noise}_i^2}} = \frac{\text{Total Source Count}}{\sqrt{\sigma_{\text{RON}}^2+\sigma_{\text{DN}}^2+\ldots}},$$

$$\sigma_{\text{RON}} = \sqrt{N_{\text{pix}}\cdot \text{RON}^2},\quad \sigma_{\text{DN}} = \sqrt{N_{\text{pix}}\cdot \text{DN}\cdot t},$$

where t is the exposure time.

Solution

From plate scale calculated in a) size of a star for typical seeing 1 arcsec

$$= \frac{1/60\ \text{arcmin}}{\left(3.44\ \text{arcmin/mm}\times 6\times 10^{-3}\ \text{mm}\right)}\ \text{pixels}$$ **[1 Mark]**

= 0.8 pixel

→ light from star is mostly contained within N_{pix}=1 pixel **[1 Mark]**

Alternative solution:
If a student gets the correct answer for 1 arcsec = 0.8 pixel but clearly explain that the chosen aperture size is N x "seeing" or a circular aperture with radius R_{Npix} (where R_{Npix} is not larger than 2x seeing) and calculate the **total number of pixels must be the closest round-up integer**.

And correctly use $\text{SNR} = \dfrac{\text{CR}\times t}{\sqrt{\text{N}_{\text{pix}}\text{RON}^2+(\text{N}_{\text{pix}}\times\text{DN}\times t)}}$

Then the student will be awarded full mark for the relevant parts.

This is also apply to part c)

From the definition of the Zero-point magnitude given in the question

$m = -2.5\log_{10}(\text{count rate}) + \text{Zero-Point}$ **[2 Mark]**

$m = 21,\ \text{ZP} = 18.5$

→ source count rate $\text{CR} = 10^{-(21-18.5)/2.5} = 0.1$ per second **[1 Mark]**

Signal-to-noise ratio

$$\text{SNR} = \frac{\text{CR}\times t}{\sqrt{\text{RON}^2 + (\text{DN}\times t)}}$$

$(\text{CR}\times t)^2 = 25(\text{RON}^2 + (\text{DN}\times t))$ **[1 Mark]**

Solve quadratic equation

$$t = \frac{25\,\text{DN} + \sqrt{(25\text{DN})^2 + 100\text{CR}^2\text{RON}^2}}{2\text{CR}^2}$$ **[1 Mark]**

$$t = \frac{\frac{25}{60} + \sqrt{(\frac{25}{60})^2 + 100(0.1)^2 10^2}}{2\times 0.1^2} = 521\text{ s} = 8.68\text{ minutes}$$ **[1 Mark]**

c) Normally when the exposure time is long and the source count is high then Poisson noise from the source is also significant. Determine the relation between SNR and exposure time in the case that the noise is dominated by Poisson noise of the source. Recalculate the minimum exposure time required to reach 21 mag with SNR=5 in part b) if Poisson noise is also taken into consideration. The Poisson noise (standard deviation) of the source is given by $\sigma_{\text{source}} = \sqrt{\text{Source Count}}$. In reality, there is also the sky background which can be important source of Poisson noise. For our purpose here, please ignore any sky background in the calculation. [6]

Solution

If we include the source Poisson noise,

$$\text{SNR} = \frac{\text{CR}\times t}{\sqrt{\text{RON}^2 + (\text{DN}\times t) + \text{CR}\times t}}$$ **[1 Mark]**

For the Poisson noise dominated case

$$\text{SNR} \approx \frac{\text{CR}\times t}{\sqrt{\text{CR}\times t}}$$ **[1 Mark]**

Hence $\text{SNR} \propto t^{\frac{1}{2}}$ **[1 Mark]**

Re-calculate exposure time

$$\text{SNR} = \frac{\text{CR} \times t}{\sqrt{\text{RON}^2 + (\text{DN} \times t) + \text{CR} \times t}},$$

$$(\text{CR} \times t)^2 = 25(\text{RON}^2 + (\text{DN+CR})t))$$ **[1 Mark]**

Solve quadratic equation

$$t = \frac{25(\text{DN+CR}) + \sqrt{25^2(\text{DN+CR})^2 + 100\text{CR}^2\text{RON}^2}}{2\text{CR}^2}$$ **[1 marks]**

$t = 666$ s = 11.1 minutes **[1 marks]**

d) The typical localisation uncertainty of the GW detector is about 100 square-degrees and we would like to cover the entire possible location of any candidate within an hour after the GW is detected. Estimate the minimum side length of the square CCD needed for each telescope in terms of the number of pixels. You may assume that the time taken for the CCD read-out and the pointing change are negligible. [6]

Solution

In 1 hr, one can observe $\frac{60}{11.1} = 5.4$ **[1 mark]** → 5 pointing (round down to the nearest integer) **[1 mark]**

Therefore, we need to cover 100 deg^2 → one pointing need to cover

$\frac{100}{(5 \times 4 \text{ telescope})} = 5 \text{ deg}^2$ **[1 mark]**

or 2.24 x 2.24 deg per pointing per telescope **[1 mark]**

calculate size of each CCD $= \frac{2.24 \text{ deg} \times 60}{(3.4 \text{ arcmin/mm})(6 \times 10^{-3} \text{ mm})}$ pixels **[1 mark]**

= 6588.2 pixels → 6589 pixels (pixel number need to be integer)

Therefore, we need minimum CCD size = 6589×6589 pixels **[1 mark]**

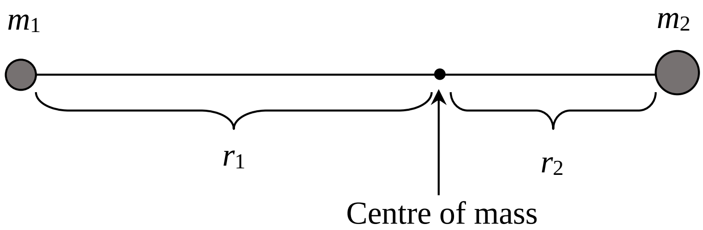
Answer 6588 pixel is also eligible for the mark

(T11) Mass of the Local Group **[50 marks]**

The dynamics of M31 (Andromeda) and the Milky Way (MW) can be used to estimate the total mass of the Local Group (LG). The basic idea is that galaxies currently in a binary system were at approximately the same point in space shortly after the Big Bang. Besides, the mass of the local group is dominated by the masses of the MW and M31. Via Doppler shifts of the spectral lines, it was found that M31 is moving toward MW with a speed of 118 km s^{-1}. This may be surprising, given that most galaxies are moving away from each other with the general Hubble flow. The fact that the M31 is moving towards MW is presumably because their mutual gravitational attraction has eventually reversed their initial velocities. In principle, if the pair of galaxies is well-represented by isolated point masses, their total mass may be determined by measuring their separation, relative velocity and the time since the universe began. Kahn and Woltjer (1959) used this argument to estimate the mass in the LG.

In this problem we will follow this argument through our calculation as follows.

a) Consider an isolated system with negligible angular momentum of two gravitating point masses m_1 and m_2 (as observed by an inertial observer at the centre of mass).



Write down the expression of the total mechanical energy (E) of this system in mathematical form connecting m_1, m_2, r_1, r_2, v_1, v_2, and the universal gravitational constant G, where v_1 and v_2 are the radial velocities of m_1 and m_2, respectively. **[5]**

Solution:

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2-\frac{Gm_1m_2}{r_1+r_2}=E.$$ Note that the total energy is negative quantity. **[5]**

Detailed Marking Scheme:

Students get 1.5 marks for each kinetic energy term.
Students get 1.0 mark for correct sign in from of the potential energy term and another 1.0 mark for the correct expression of the potential energy.

b) Re-write the equation in a) in terms of r, v, μ, M, and G, where $r \equiv r_1 + r_2$ is the separation distance between m_1 and m_2, v is the changing rate of the separation distance,

$\mu \equiv \dfrac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the system, and $M \equiv m_1 + m_2$ is the total mass of the system. [10]

Solution:

From $m_1 r_1 = m_2 r_2$ and $r_1 + r_2 = r$, we have $r_1 = \dfrac{m_2}{m_1 + m_2} r$ and $r_2 = \dfrac{m_1}{m_1 + m_2} r$.

Substituting them into the solution obtained in a), we get

$$\frac{1}{2} v^2 - \frac{G(m_1 + m_2)}{r} = \frac{E}{m}$$

$$v^2 - \frac{2GM}{r} = \frac{2E}{m}$$

Therefore, $E = \dfrac{\mu}{2}\left(v^2 - \dfrac{2GM}{r}\right)$ [10]

Detailed Marking Scheme:

Students get 2.0 marks for using the conservation of momentum or center of mass to get $m_1 r_1 = m_2 r_2$.

Students get 1.0 mark for each of the relationship $r_1 = \dfrac{m_2}{m_1 + m_2} r$ or $r_2 = \dfrac{m_1}{m_1 + m_2} r$. (total 2.0 marks)

Students get 1.0 mark each for the relationships between v and v_1, and between v and v_2. (total 2.0 marks)

For correct substitution of suitable variables into the expression for E, students get 2.0 marks.

Final expression for E is worth 2.0 marks.

c) Show that the equation in b) yields

$$v^2 = (2GM)\left(\frac{1}{r} - \frac{1}{r_0}\right),$$ where r_0 is a new constant.

Find r_0 in terms of μ, M, G and E. [5]

Solution:

From b), we have $E = \frac{\mu}{2}\left(v^2 - \frac{2GM}{r}\right)$ or $v^2 - \frac{2GM}{r} = \frac{2E}{m}$

$$v^2 = \frac{2GM}{r} + \frac{2E}{m}$$

$$= (2GM)\left\{\frac{1}{r} + \frac{1}{\left(\frac{GM\mu}{E}\right)}\right\}$$

$$= (2GM)\left\{\frac{1}{r} - \frac{1}{r_0}\right\}, \text{ where } r_0 \equiv -\frac{GM\mu}{E}$$

Thus

$$r_0 = -\frac{GMm}{E}$$ [5]

Detailed Marking Scheme:

Students get 4.0 marks for correct algebra and obtain the expression in the correct form.

By correctly identifying that $r_0 = -\frac{GMm}{E}$, students get 1.0 mark.

The solution of the equation in b) is given below in parametric form, under the initial condition $r = 0$ at $t = 0$:

$$r(\theta) = \frac{r_0}{2}(1 - \cos\theta),$$

$$t(\theta) = \left(\frac{r_0^3}{8GM}\right)^{\frac{1}{2}}(\theta - \sin\theta),$$

where θ is in radians.

d) From the above parametric equations, show that an expression for $\frac{vt}{r}$ is

$$\frac{vt}{r} = \frac{(\sin q)(q - \sin q)}{(1 - \cos q)^2}.$$ [10]

Solution:

Substituting $r(\theta)$ in the equation of part c):

$$v^2 = 2GM\left(\frac{1}{r}-\frac{1}{r_0}\right) = 2GM\left(\frac{1}{\frac{1}{2}r_0\left(1-\cos\theta\right)}-\frac{1}{r_0}\right) = \frac{2GM}{r_0}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$$

$$v^2 = \frac{2GM}{r_0}\frac{(1-\cos^2\theta)}{(1-\cos\theta)^2} = \frac{2GM}{r_0}\frac{\sin^2\theta}{(1-\cos\theta)^2} \quad\Rightarrow\quad v(\theta) = \left(\frac{2GM}{r_0}\right)^{1/2}\frac{\sin\theta}{1-\cos\theta}$$

Combining $v(\theta)$, $t(\theta)$, and $r(\theta)$:

$$\frac{v(\theta)t(\theta)}{r(\theta)} = \frac{\left(\frac{2GM}{r_0}\right)^{1/2}\frac{\sin\theta}{1-\cos\theta}\cdot\left(\frac{r_0^3}{8GM}\right)^{1/2}(\theta-\sin\theta)}{\frac{r_0}{2}(1-\cos\theta)} = \frac{2}{r_0}\left(\frac{2GMr_0^3}{r_0 8GM}\right)^{1/2}\frac{(\sin\theta)(\theta-\sin\theta)}{(1-\cos\theta)^2}$$

$$\frac{vt}{r} = \frac{(\sin\theta)(\theta-\sin\theta)}{(1-\cos\theta)^2}$$

Alternative solution:

From $r(q) = \frac{r_0}{2}(1 - \cos q)$, we have

$$v = \left(\frac{r_0}{2}\sin\theta\right)\omega$$

From $t(\theta) = \left(\frac{r_0^3}{8GM}\right)^{1/2}(\theta-\sin\theta)$, we have

$$1 = \left(\frac{r_0^3}{8GM}\right)^{1/2}(1-\cos\theta)\omega$$

$$\therefore \frac{vt}{r} = \frac{\left(\frac{r_0}{2}\sin\theta\right)\left(\frac{r_0^3}{8GM}\right)^{1/2}(\theta-\sin\theta)}{\left(\frac{r_0^3}{8GM}\right)^{1/2}(1-\cos\theta)\frac{r_0}{2}(1-\cos\theta)} = \frac{(\sin\theta)(\theta-\sin\theta)}{(1-\cos\theta)^2}. \qquad [10]$$

Detailed Marking Scheme:

Student gets 3.0 marks for substituting $r(\theta)$ in v^2 and obtaining $\frac{2GM}{r_0}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$.

Student gets 3.0 marks for using trigonometry to simplify the expression for $v(\theta)$.

For the correct derivation of the final answer, student gets 4.0 marks.

e) Now we consider m_1 and m_2 as the MW and M31 respectively. The current values of v and r are $v = -118 \text{ km s}^{-1}$ and $r = 710 \text{ kpc}$, and t may be taken to be the age of the Universe (13700 million years). Find θ using numerical iteration. [10]

Solution:

$$v = -118 \text{ km s}^{-1} = -118 \times 10^3 \text{ m s}^{-1}$$

$$r = 710 \text{ kpc} = 2.19 \times 10^{22} \text{ m}$$

$$t = 13700 \times 10^6 \text{ years} = 4.3233 \times 10^{17} \text{ s}$$

$$\therefore \frac{vt}{r} = -\frac{118 \times 10^3 \times 4.3233 \times 10^{17}}{2.1908 \times 10^{22}} = -2.33 = \frac{(\sin\theta)(\theta - \sin\theta)}{(1-\cos\theta)^2}$$ [5]

The negative value of the left-hand side of this equation implies that θ is greater than π.

θ	$(\sin\theta)(\theta-\sin\theta)/(1-\cos\theta)^2$
$200^\circ = 3.49$ radians	-0.348
$210^\circ = 3.67$ radians	-0.598
$240^\circ = 4.19$ radians	-1.95
$244^\circ = 4.26$ radians	-2.24
$245^\circ = 4.28$ radians	-2.32 ←
$246^\circ = 4.29$ radians	-2.40
$250^\circ = 4.36$ radians	-2.77

This gives the value of

$245^\circ = 4.28$ radians

[5]

Detailed Marking Scheme:

Students get 5.0 marks if numerical value and sign (2.5 marks for each) are both correct.

Students get 5.0 marks if the obtained angle (θ) is within +/- 1 degree (or equivalent in radian), or 2.0 marks for answer within +/- 3 degrees (or equivalent in radian).

f) Use the value of θ in e) to obtain the value of $r_{\max}$. Hence also obtain the value of M in solar masses. [10]

Solution:

From $r(\theta)=\frac{r_0}{2}(1-\cos\theta)$, we have $r_{\max}=r_0$, and hence

$$r_{\max}=\frac{2r(\theta)}{1-\cos\theta}=\frac{2\times 710\text{ kpc}}{1-\cos 245^\circ}=998\text{ kpc} \qquad [5]$$

$\left(\begin{array}{l}\text{This is the maximum separation between the Milky-Way Galaxy and}\\ \text{Andromeda Galaxy; they started to move towards each other afterwards.}\end{array}\right)$

The value of M can be calculated from the relation $t(\theta)=\left(\frac{r_0^3}{8GM}\right)^{\frac{1}{2}}(\theta-\sin\theta)$.

$$M=\left(\frac{r_{\max}^3}{8G}\right)\left\{\frac{\theta-\sin\theta}{t(\theta)}\right\}^2.$$

From the maximum separation of $r_{\max}=998\text{ kpc}=3.0795\times10^{22}\text{ m}$,
$\theta=245^\circ=4.276\text{ radians}$, $t=t(\theta)=13700\times10^6\text{ years}=4.3233\times10^{17}\text{ s}$, and
$G=6.67\times10^{-11}\text{ N m}^2\text{kg}^{-2}$, we obtain

$$\therefore M=7.86\times10^{42}\text{ kg}=\frac{7.86\times10^{42}}{1.99\times10^{30}}M_{\odot}=3.95\times10^{12}M_{\odot} \qquad [5]$$

The estimated number of stars, most of which are of less than a solar mass, for Andromeda and Our Galaxy are 4×10^{11} and 2×10^{11} respectively. This implies that most of mass of the system is DARK.

Detailed Marking Scheme:

First part which is counting for 5.0 marks for determining $r_{\max}$ is detailed as follow:
Students get 2.0 marks for realizing that $r_{\max}=r_0$.
Students get 1.0 mark by re-arranging $r_{\max}$ in term of $r(\theta)$ and θ.
Students get 1.0 mark if they substitute the correct values given in the question.
Students get 1.0 mark for correct numerical answer.

Second part that is worth 5.0 marks for determining M is detailed as follow:
Students get 2.0 marks for successfully expressing M in term of known variables.

Students get 1.0 mark by substituting the correct numerical values given in the question into the expression from previous step.
Students get 2.0 marks for obtaining the correct answer in requested unit.

Note that

$$v = \frac{d}{dt} r = +(2GM)^{1/2} \left(\frac{r_0 - r}{r_0 r} \right)^{1/2} \text{ for } r_0 > r$$

$$\int (2GM)^{1/2} dt = +\int \left(\frac{r_0 r}{r_0 - r} \right)^{1/2} dr$$

Putting $\frac{r}{r_0} \equiv \sin^2 \phi$

$$t = \left(\frac{r_0^3}{8GM} \right)^{1/2} \{2\phi - \sin 2\phi\} + D$$

We choose the initial condition such that m_1 and m_2 are close together, that is that $r = 0$ at $t = 0$. This implies that $\phi = 0$ at $t = 0$, hence D must be zero.

$$t = \left(\frac{r_0^3}{8GM} \right)^{1/2} (\theta - \sin\theta), \ \theta \equiv 2\phi .$$

And from $r = r_0 \sin^2 \phi = \frac{1}{2} r_0 (1 - \cos 2\phi),$ we have:

$$r = \frac{1}{2} r_0 (1 - \cos\theta).$$

Equation (5) and (6) form the parametric solution of equation (4).

$$r_{\max} = r(\theta = \pi) = r_0$$

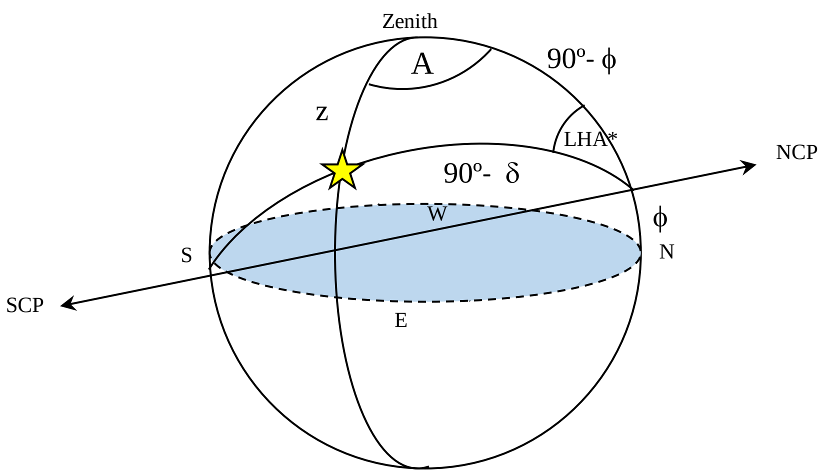
(T12) Shipwreck **[40 mark]**

You are shipwrecked on an island. Fortunately, you are still wearing a watch that is set to Bangkok time. You also have a compass, an atlas and a calculator. You are initially unconscious, but wake up to find it has recently become dark. Unfortunately it is cloudy. An hour or so later you see Orion through a gap in the clouds. You estimate that the star "Rigel" is about 52.5º above the horizon and with your compass you find that it has an astronomical azimuth of 109º. Your watch says 01:00 on the 21st November 2017. You happen to remember from your astronomy class that Greenwich Sidereal Time (GST) at 00h UT 1st January 2017 is about 6h 43min and that R.A. and Dec. of Rigel are 5h 15min and $-8^\circ\ 11'$, respectively.

a) Find the Local Hour Angle (LHA) of Rigel. [10]

Solution:

Draw a correct celestial sphere [4.0]



$$\frac{\sin \text{LHA*}}{\sin z} = \frac{\sin \text{A}}{\sin(90^\circ - \delta)}$$ [2.0]

$$\text{LHA}^* = \sin^{-1}\left(\frac{\sin \text{A} \sin z}{\sin(90^\circ - \delta)}\right)$$

$$= \sin^{-1}\left(\frac{\sin 109^\circ \sin 37.5^\circ}{\sin 98.18^\circ}\right)$$ [2.0]

$$\text{LHA}^* = 35.56^\circ = 2\text{h } 22 \text{ min}$$

$$\text{LHA} = 24\text{h} - 2\text{h } 22 \text{ min} = 21\text{h } 38 \text{ min}$$ [2.0]

b) Find the current Greenwich sidereal time (GST). [10]

Solution:

Bangkok time of 01:00 21st November is Universal Time (UT) 18:00 20th November

Day number + Time = 323 day 18 h [1.0]

GST = GST of 1st Jan + The angle that Y moves away from 1st Jan [2.0]

$$GST = 6h\ 43min + \left(323\frac{18}{24}\text{days}\right)\left(\frac{1}{365.2422}\right) \times 24\frac{h}{day} + \left[18h \times \left(\frac{24}{23.9344}\right)\right]$$ [5.0]

GST = 6h 43min + 21h 16 min + 18h 3min

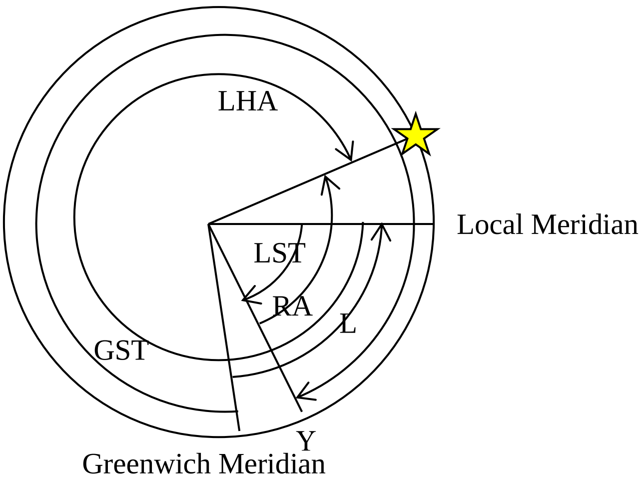
GST = 46h 2min

GST = 21h 58min [2.0]

Any other correct alternative methods that provide the correct answer also acceptable.

c) Find the longitude of the island. [5]

Solution:



$24h - LHA = RA - LST$

$LST = RA - 24h + LHA$

$L = RA + LHA - GST$

$L = 5h\ 15min + 21h\ 38\ min - 21h\ 58min = 4h\ 55min$ [3.0]

$\mathrm{L} = 73.75^{\circ}\,\mathrm{E}$ [2.0]

d) Find, accurately to the nearest arcminute, the Latitude of the island. [15]

Solution:

$$\cos(90^{\circ} - \delta) = \cos z \cos(90^{\circ} - \phi) + \sin z \sin(90^{\circ} - \phi) \cos \mathrm{A}$$

$$\sin \delta = \sin(h) \sin \phi + \cos(h) \cos \mathrm{A} \cos \phi$$ [5.0]

$$\alpha = \sin \delta, \ \beta = \sin(h), \ \gamma = \cos(h) \cos \mathrm{A}, \ x = \sin \phi$$

$$\alpha = \beta x + \gamma \sqrt{1 - x^2}$$

$$(\beta^2 + \gamma^2) x^2 - 2\alpha\beta x + (\alpha^2 - \gamma^2) = 0$$ [1.0]

$$x = \frac{\alpha\beta \pm \gamma \sqrt{\gamma^2 + \beta^2 - \alpha^2}}{\beta^2 + \gamma^2}$$ [1.0]

$$\sin \phi = \frac{\sin \delta \sin(h) \pm \cos(h) \cos \mathrm{A} \sqrt{(\cos(h) \cos \mathrm{A})^2 + \sin^2(h) - \sin^2 \delta}}{\sin^2(h) + (\cos(h) \cos \mathrm{A})^2}$$ [1.0]

$$\phi = -24.05^{\circ}, \ 4.01^{\circ}$$ [2.0]

For $\phi = -24.05^{\circ} \rightarrow \mathrm{A} = 71.0^{\circ}$ [2.0]

For $\phi = 4.01^{\circ} \rightarrow \mathrm{A} = 109^{\circ}$ [2.0]

The Latitude of the island is $4^{\circ}1'\,\mathrm{N}$ [1.0]

Alternative Solutions

b)

Solution:

$-$LMT (Local Mean Time) $\rightarrow$ UT [1.0]

Calculate sidereal day using given tropical year

$$\gamma = \frac{\text{Sidereal time}}{\text{Solar time}} = \frac{24^h \times \left(1 + \frac{1}{T_{\text{tropic}}}\right)}{24^h}$$

$$= 1 + \frac{1}{T_{\text{tropic}}} = 1.002737909$$ [3.0]

On the 21st November 2017, GST can be calculated from GST on 1st January 2017

Day number + Time = 323 d 18 h [2.0]

$$\text{GST} = \text{GST}_0 + \Delta\theta = \text{GST}_0 + \gamma\,\Delta\text{T}$$ [2.0]

$$\Delta\theta = \left(323^{\text{d}}\ \tfrac{18}{24}\right)1.002737909$$

$$= 324^{\text{d}}.6363981 = 15^{\text{h}}16^{\text{m}}24^{\text{s}}.8$$

$$\text{GST} = \text{GST}_0 + 15^{\text{h}}16^{\text{m}}24^{\text{s}}.8$$

$$= 6^{\text{h}}43^{\text{m}} + 15^{\text{h}}16^{\text{m}}24^{\text{s}}.8$$

$$= 21^{\text{h}}59^{\text{m}}24^{\text{s}}.8$$ [2.0]

d)

Solution:

From Cosine law:

$$\sin\delta = (\cos z)\sin\phi + (\sin z\cos A)\cos\phi$$

$$\cos z = (\sin\delta)\sin\phi + \left(\cos\delta\cos\text{LHA}^*\right)\cos\phi$$

[5.0]

Correct calculation steps that can lead to the final answers. [5.0]

Final answers

$\sin\phi = 0.0698$

$\phi = 4.00°$

[5.0]

OR

$\cos\phi = 0.9976$

$\phi = 4.00°$

(T13) Exomoon **[60 marks]**

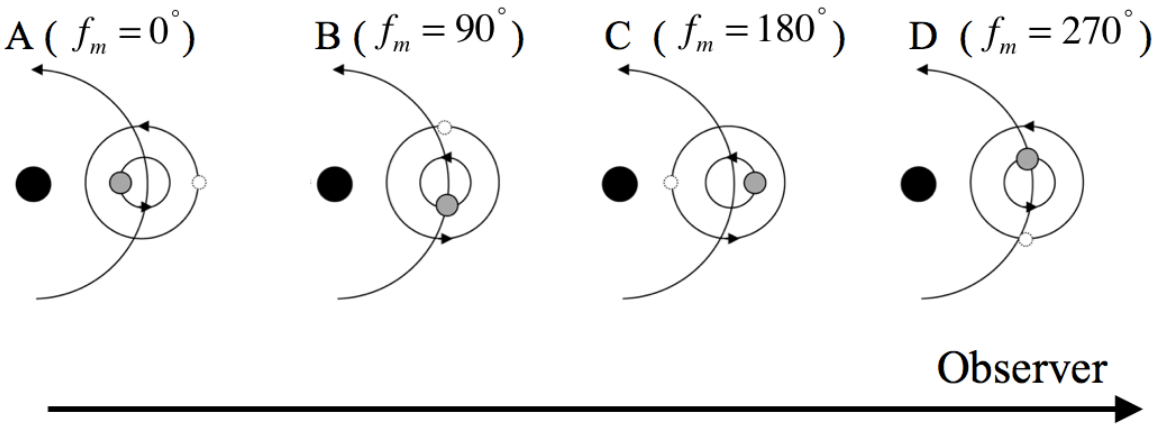
Exomoons are natural satellites of exoplanets. The gravitational influence of such a moon will affect the position of the planet relative to the planet-moon barycentre, resulting in Transit Timing Variations (σ_{TTV}, TTVs) as the observed transit of the planet occurs earlier or later than the predicted time of transit for a planet without a moon.

The motion of the planet around the planet-moon barycentre will also induce Transit Duration Variations (σ_{TDV}, TDVs) as the observed transit duration is shorter or longer than the predicted transit duration for a planet without a moon.

We will consider edge-on circular orbits with the following parameters

- M_P is the planet mass
- M_m is the moon mass
- P_P is the planet-moon barycentre's period around the host star
- P_m is the moon's period around the planet
- a_p is the distance of the planet-moon barycentre to the star
- a_m is the distance of the moon to the planet-moon barycentre
- f_m is the moon phase, $f_m = 0^\circ$ when the moon is in opposition to the star
- τ is the mean transit duration of the planet (as if it has no moon)

We will consider only orbit of a prograde moon with an orbit in the same plane as the planet's orbit. Example phases of the moon, as observed by distant observers, are shown in the figure below.



Phase of the moon.
Black, grey and white circles represent the star, planet and moon, respectively.

a. We define $\sigma_{TTV} \equiv t_m - t$ where t is the predicted transit time without the moon, and t_m is the observed transit time with the moon. Show that

$$\sigma_{TTV} = \left[\frac{a_m M_m P_p}{2\pi a_p M_p}\right]\sin(f_m)$$

A positive value of σ_{TTV} indicates that the transit occurs later than the predicted time of transit for a planet without a moon. [10]

Solution:

From centre of mass for planet-moon system, the distance of the moon to the planet-moon barycenter can be written as,

$$a_m = \frac{M_p}{M_m} a_{pb}$$ **[2.0]**

where a_{pb} is distance of the planet to the planet-moon barycentre.

For the observers on the Earth, projection distance of the planet to the planet-moon barycenter is

$$a_{proj} = a_{pb} \sin(f_m)$$ **[3.0]**

From angular velocity, TTV signal can be calculated from

$$\sigma_{TTV} \equiv t_m - t$$

$$\sigma_{TTV} = \frac{\theta_{trans}}{\omega_p}$$ **[1.0]**

$$\sigma_{TTV} = \left(\frac{a_{proj}}{a_p}\right) \Big/ \left(\frac{2\pi}{P_p}\right)$$ **[2.0]**

$$\sigma_{TTV} = \frac{a_{proj} P_p}{a_p 2\pi}$$

$$\sigma_{TTV} = \left[\frac{a_m M_m P_p}{2\pi a_p M_p}\right] \sin(f_m)$$ **[2.0]**

b. $\sigma_{TDV} \equiv \tau_m - \tau$ where τ is the predicted transit duration without the moon, and τ_m is the observed transit duration with the moon. We can assume that the planet's velocity around the star is much bigger than the moon's velocity around the planet-moon barycentre, and also the moon does not change phase during the transit. Show that

$$\sigma_{TDV} = \tau \left[\frac{P_p M_m a_m}{P_m M_p a_p}\right] \cos(f_m)$$

A positive value of σ_{TDV} indicates that the transit duration is longer than the predicted transit duration without a moon. [13]

Solution:

The velocity of planet-moon barycenter around the star is

$$v_p = \frac{2\pi}{P_p} a_p \quad \textbf{[1.0]}$$

The velocity of planet around the planet-moon barycentre is

$$v_{pb} = \frac{2\pi}{P_m} a_{pb} \quad \textbf{[1.0]}$$

Therefore, the transverse velocity of planet around the planet-moon barycentre is

$$v_{trans} = -v_{pb}\cos(f_m) \quad \textbf{[3.0]}$$

$$v_{trans} = -\frac{2\pi M_m}{P_m M_p} a_m \cos(f_m)$$

Average transit duration can be written as

$$\tau = \frac{D}{v_p} \quad \textbf{[2.0]}$$

where D is the distance of the planet has to cross in order to complete the transit. The transit duration with exomoon can be written as

$$\tau_m = \frac{D}{v_p + v_{trans}} \quad \textbf{[2.0]}$$

Therefore, the TDV signal is

$$\sigma_{TDV} \equiv \tau_m - \tau = \frac{\tau v_p}{v_p + v_{trans}} - \tau \quad \textbf{[1.0]}$$

$$\sigma_{TDV} = \tau\left[\frac{-v_{trans}}{v_p + v_{trans}}\right]$$

In case $v_p >> v_{trans}$

$$\sigma_{TDV} = -\tau\left[\frac{v_{trans}}{v_p}\right] \quad \textbf{[2.0]}$$

$$\sigma_{TDV} = \tau\left[\frac{P_p M_m a_m}{P_m M_p a_p}\right]\cos(f_m) \quad \textbf{[1.0]}$$

An exoplanet is observed transiting a main-sequence solar type star (1 $M_{\odot}$, 1 $R_{\odot}$, Spectral class: G2V).The planet has an edge-on circular orbit with a period of 3.50 days. From the observational data, the planet has a mass of 120 $M_{\oplus}$ of and a radius of 12 $R_{\oplus}$. The observed relation between σ_{TTV}^2 and σ_{TDV}^2 can be written as

$$\sigma_{TDV}^2 = -0.7432\sigma_{TTV}^2 + 1.933\times10^{-8}\ \text{days}^2$$

c. Assume that the moon's mass is much smaller than the planet's mass. Find the mean transit duration of the planet (τ) in days. [6]

Solution:

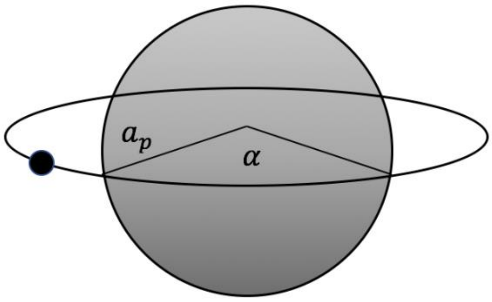
From Kepler's third's law, assume that the moon mass is much smaller than the planet mass, the distance of the planet-moon barycentre to the star is

$$P^2 = \frac{4\pi^2}{G(M_* + M_p)} a_p^3$$ **[1.0]**

$$(3.5 \times 86400)^2 = \frac{4\pi^2}{6.67 \times 10^{-11} \times (1.99 \times 10^{30} + 120 \times 5.98 \times 10^{24})} a_p^3$$

$$a_p = 6.75 \times 10^9 \text{ m}$$

The mean transit duration of the planet can be calculated from the transit duration of exoplanet without exomoon.



$$\tau = P\frac{\alpha}{2\pi}$$ **[1.0]**

$$\tau = \frac{P}{2\pi} \times 2\sin^{-1}\left(\frac{R_* + R_p}{a_p}\right)$$ **[2.0]**

$$\tau = \frac{3.5}{\pi}\sin^{-1}\left(\frac{6.96 \times 10^8 + 12 \times 6.38 \times 10^6}{6.75 \times 10^9}\right) \text{ days}$$

$$\tau = 0.128 \text{ days}$$ **[2.0]**

d. Find the moon's period (P_m) in days [7]

Solution:

The TTV signal is 90 degrees out of phase with the TDV signal in edge-on circular orbit system. Therefore, the relation between the TTV and TDV signals is,

$$\sigma_{TDV}^2 = -\left(\frac{2\pi\tau}{P_m}\right)^2 \sigma_{TTV}^2 + \tau^2\left(\frac{a_m M_m P_p}{a_p M_p P_m}\right)^2$$ **[3.0]**

The observed relation between σ^2_{TTV} and σ^2_{TDV} can be written as

$$\sigma^2_{TDV} = -0.7432\sigma^2_{TTV} + 1.933\times10^{-8}\ \text{days}^2$$

Therefore

$$-\left(\frac{2\pi\tau}{P_m}\right)^2 = -0.7432 \qquad \textbf{[2.0]}$$

$$P_m = 0.933\ \text{days} \qquad \textbf{[2.0]}$$

e. Estimate the distance of the moon to the planet-moon barycentre (a_m) in units of Earth radii. Also find the moon mass (M_m) in units of Earth mass. [7]

Solution:

From Kepler third law, assuming that moon is small and can be neglected, the distance of the moon to the planet-moon barycentre is

$$P_m^2 = \frac{4\pi^2}{GM_p}a_m^3 \qquad \textbf{[1.0]}$$

$$(0.933\times86400)^2 = \frac{4\pi^2}{6.67\times10^{-11}\times(120\times5.98\times10^{24})}a_m^3$$

$$a_m = 1.99\times10^8\ \text{m} = 31.2R_\oplus \qquad \textbf{[2.0]}$$

From the observed relation between σ^2_{TTV} and σ^2_{TDV}, the moon mass is

$$\tau^2\left(\frac{a_m M_m P_p}{a_p M_p P_m}\right)^2 = 1.933\times10^{-8}\ \text{days}^2 \qquad \textbf{[2.0]}$$

$$(0.128)^2\left(\frac{(1.99\times10^8)\times M_m\times(3.5)}{(6.75\times10^9)\times(120)\times(0.933)}\right)^2 = 1.933\times10^{-8}\ \text{days}^2$$

$$M_m = 1.18M_\oplus \qquad \textbf{[2.0]}$$

f. The Hill sphere is a region around a planet within which the planet's gravity dominates. The radius of the Hill sphere can be written as

$$R_h = a_p\sqrt[3]{\frac{M_p}{xM_*}}$$

where M_* is the host star mass.

Find the value of the constant x (Hint: for a massive host star, the radius of the Hill sphere of the system is approximately equal to the distance between the planet and the Lagrange point L_1or L_2). Then find the radius of the Hill sphere of thisplanetary system in units of Earth radii. [11]

Solution:

For planet-moon system, the angular velocity of barycenter of planet-moon system is

$$F_{centrifugal-bary} = F_{star-bary}$$

$$(M_p + M_m)\omega_p^2 a_p = \frac{GM_*(M_p + M_m)}{a_p^2}$$

$$\omega_p^2 = \frac{GM_*}{a_p^3}$$ **[2.0]**

For exomoon at Lagrange point L_1or L_2,

$$F_{centrifugal-moon} = F_{star-moon} \pm F_{planet-moon}$$ **[1.0]**

$$M_m W_p^2 (a_p \pm a_m) = \frac{GM_* M_m}{(a_p \pm a_m)^2} \pm \frac{GM_p M_m}{(a_{pb} + a_m)^2}$$ **[3.0]**

$$\frac{M_*(a_p \pm a_m)}{a_p^3} = \frac{M_*}{(a_p \pm a_m)^2} \pm \frac{M_p}{(a_{pb} + a_m)^2}$$

Since $M_p >> M_m$ and $a_{pb} << a_m$

$$\frac{M_*(a_p \pm a_m)}{a_p^3} = \frac{M_*}{(a_p \pm a_m)^2} \pm \frac{M_p}{a_m^2}$$ **[1.0]**

$$M_*(a_p \pm a_m)^3 a_m^2 = M_* a_p^3 a_m^2 \pm M_p a_p^3 (a_p \pm a_m)^2$$

$$M_*(\pm 3a_p^2 a_m^3 + 3a_p a_m^4 \pm a_m^5) = \pm M_p a_p^3 (a_p \pm a_m)^2$$

For $a_p >> a_m$, $M_*(3a_p^2 a_m^3) \approx M_p a_p^5$ **[1.0]**

$$a_m^3 = \frac{M_p}{3M_*} a_p^3$$

$$a_m = a_p \sqrt[3]{\frac{M_p}{3M_*}}$$

$$x = 3$$ **[1.0]**

The radius of the Hill sphere of this planetary system is

$$R_h = a_p \sqrt[3]{\frac{M_p}{3M_*}}$$

$$R_h = 6.75 \times 10^9 \times \sqrt[3]{\frac{120 \times 5.98 \times 10^{24}}{3 \times 1.99 \times 10^{30}}} \text{ m}$$

$R_h = 3.33\times10^{8}\ \text{m} = 52.2R_{\oplus}$ **[2.0]**

g. The Roche limit is the minimum orbital radius at which a satellite can orbit without being torn apart by tidal forces and take the Roche limit as

$$R_r = 1.26R_p\sqrt[3]{\frac{\rho_p}{\rho_m}}$$

where ρ_p and ρ_m are the density of the planet and moon, respectively. R_p is planet radius. Assuming that the moon is a rocky moon with Earth's density, find the Roche limit of the system. [3]

Solution:

From Roche limit equation

$$R_r = 1.26R_p\sqrt[3]{\frac{\rho_p}{\rho_m}}$$

$$R_r = 1.26\times12\times\sqrt[3]{\frac{120/(12)^3}{1}}R_{\oplus}$$

$$R_r = 6.21R_{\oplus}$$ **[3.0]**

h. Does the moon have a stable orbit? [3]

Solution:

Stable orbit because the distance between the moon and planet-moon barycentre is between Roche limit and Hill sphere radius ($R_r < a_m < R_H$) **[3.0]**

Mark only if students get the answers for e) & g) &h).
Students obtain full mark if answer "YES" for $R_r < a_m < R_H$, and vice versa,