Instructions

- Enter your *Registration Number* here: **CMIPG ID** | | | | | | | | |

  Enter your *Examination Centre* here: | | | | | | | | | | | | |

  - The time allowed is 3 hours.
  - Total Marks: 100
  - Each question carries 20 marks.
- Answer all questions.
- *Rough Work:* The coloured blank pages are to be used for rough work only.

For office use only

1	
2	
3	
4	
5	
Total	

CMI Ph. D. Physics Entrance Exam 2016

(1) A particle is constrained to move on a parabola $y = \alpha x^2$ in the plane where α is a constant of dimension inverse length. Let there be a constant gravitational force in negative z direction.

(a) Write down the suitable generalized co-ordinates for this system. [1 Mark]
(b) Write down the Lagrangian and obtain Lagrangian equations of Motion. [1 Mark]
(c) Find the equilibrium position of the particle and write the equation for small ocscillations about this equilibrium. [1 Mark]
(d) By solving the above equation show that particle executes simple harmonic motion about the equilibrium position. [2 Marks]

(2) Consider three Newtonian particles of masses m_1, m_2, m_3 which interact with each other only through Newtonian gravitational force. Let their position vectors be $\vec{r}_1, \vec{r}_2$, and $\vec{r}_3$ respectively with respect to the center of mass of the system.
(a) Write down the equations of motion for the system. [4 Marks]
(b) Assuming that the distance d between any pair of masses is equal and is in fact constant, show that the system rotates with a constant angular velocity in it's plane and about the center of mass. [6 Marks]

(3) Consider a uniform hoop of mass M and radius R which hangs in a vertical plane supported by a hinge at a point on the hoop as shown in the figure. Calculate the natural frequency of small oscillations. [5 Marks]

(4) Recall that under a gauge transformation in magnetostatics, the vector potential is transformed to $\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) - \nabla\chi(\mathbf{r})$ where $\chi(\mathbf{r})$ is a scalar function. [20 Marks]

(a) Show that Ampere's law (in differential form) takes the same form for both $\mathbf{A}$ and $\mathbf{A}'$. [2 Marks]

(b) Give the differential equation that χ must satisfy so that $\mathbf{A}'$ is in Coulomb gauge. Write an integral expression for its solution, assuming $\mathbf{A} \to 0$ sufficiently fast as $|\mathbf{r}| \to \infty$. Proceed by first writing Poisson's equation for the electrostatic potential $\phi(\mathbf{r})$ due to a localized charge distribution $\rho(\mathbf{r})$ and its solution via an integral. Give an appropriate analogy between the two problems. [5 Marks]

(c) Find the magnetic field $\mathbf{B}(\mathbf{r})$ represented by the vector potential $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{b})$ where $\mathbf{b}$ is a constant vector. Hint: For two vector fields we have $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$. [5 Marks]

(d) Find whether the above vector potential $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{b})$ is in Coulomb gauge or not. Hint: $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$. [4 Marks]

(e) Suppose the constant vector $\mathbf{b} = b_0 \hat{z}$ points along the z axis. Find the vector potential $\mathbf{A}$. Roughly plot the resulting vector field $\mathbf{A}$ in the $x - y$ plane. [4 Marks]

(5)

(a) Consider a particle of mass m in a harmonic oscillator potential of frequency ω. Suppose the particle is in the state

$$\mid \alpha >= \sum_{n=0}^{\infty} c_n \mid n >$$

where

$$c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$$

and α is a complex number. Show that $\mid \alpha >$ is an eigenstate of the annihilation operator $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$. **[3]**

(b) Show that the probability $P_n(\alpha)$ of finding the state $\mid \alpha >$ to have the value n when the operator $\hat{a}^\dagger \hat{a}$ is measured is given by the Poisson distribution:

$$P_n(\alpha) = e^{-|\alpha|^2} \frac{\mid \alpha \mid^{2n}}{n!}$$

[2]

(c) Show that in the state $\mid \alpha >$, $< x >= \sqrt{\frac{2\hbar}{m\omega}}\, \mathbf{Re}(\alpha)$ and $< p >= \sqrt{2m\hbar\omega}\, \mathbf{Im}(\alpha)$. **[2]**

(d) Show that the state $\mid \alpha >$ is a minimum uncertainty state *i.e.*

$$\Delta x \Delta p = \frac{\hbar}{2}$$

[3]

(e) If at a time $t = 0$, the state is $\mid \psi(0) >=\mid \alpha >$, show that at a later time

$$\mid \psi(t) >= e^{\frac{-i\omega t}{2}} \mid \alpha e^{-i\omega t} >$$

What is the origin of the phase factor? **[3]**

(f) From (c) and (e) show that as functions of time, $< \hat{x} >$ and $< \hat{p} >$ follow the classical trajectory of the harmonic oscillator. **[2]**

(g) Starting from $\hat{a} \mid \alpha >= \alpha \mid \alpha >$, show that the position space wave function for this state is $\psi_\alpha(x) = e^{i<\hat{p}>x/\hbar} u_0(x- < \hat{x} >)$, where $u_0(x)$ is the ground state gaussian function. **[5]**

(6) Consider a system of N free spin half particles of mass m obeying Fermi Dirac statistics at temperature T=0 and occupying a volume V.

(a) Find the Fermi energy E_F of the system. [3 Marks]

(b) How many particles have energy $E \leq E_F$? [3 marks]

(c) Calculate the average value of U^3 where U is the speed of the particle. [4 marks]

(7). Consider a ideal gas having N distinguishable particles occupying a volume V. Consider a small volume V_1.

(a) What is the probability that the volume V_1 contains r particles? [3 marks]
(b) In the limit of large N show that r is maximum for $r_0 = N V_1/V$. [3 marks]
(c) Show that the probability distribution of $r - r_0$ is a gaussian [4 marks]
You can use the Stirling's approximation $\ln \mathrm{N}! = \mathrm{N} \ln \mathrm{N} - \mathrm{N}$ for the parts (b),(c) .

(8) Consider the integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$

(a) Define $Z = e^{i\theta}$ and express the integrand as a function of Z. [2 Marks]
(b) Identify the singularities of the above function in the complex Z plane. [2 Marks]
(c) Identify the contour of integration (in the complex Z plane) and perform the integral using the residue theorem. [6 Marks]

(9) Given

$$\begin{aligned}\lambda(t) &= 1-|t|, \quad |t| \leq 1 \\ &= 0, \quad \text{otherwise.}\end{aligned}$$

(a) Plot $\lambda(t)$ [2 Marks]

(b) Find the Fourier transform $\tilde{\lambda}(\omega)$ of $\lambda(t)$. [6 Marks]

(c) Plot $\tilde{\lambda}(\omega)$. [2 Marks]