[1] The partition function for a statistical mechanical system, at temperature T, is defined by the equation

$$Z = \sum_i e^{-\beta E_i}$$

where the index i labels a microscopic state with energy E_i and $\beta = 1/(k_B T)$, k_B being the Boltzmann constant.
For a paramagnet, the energy of a microscopic state i of N spins is given by

$$E_i = -\mu_B H(\sigma_1 + \sigma_2 + \sigma_3 \cdots + \sigma_N)$$

where H is the magnetic field, μ_B, the magnetic moment and $\sigma = \pm 1$.

(a) Show that the partition function for the above system is given by the expression

$$Z(T, H, N) = [2\cosh(\mu_B \beta H)]^N$$

(b) Find the Helmholtz free energy, F, for the system.

(c) Find the Entropy, S, of the system.

(d) Obtain an expression for the specific heat at constant field H from the expression for S.

(e) If the energy of the microstate changes by the addition of a constant independent of the state,

$$E_i \to E_i + C,$$

how do the partition function (Z), average energy (E), free energy (F), and the magnetization (M) change?

[2] Consider a particle of mass μ in three dimensions. It is subjected to an attractive central force of magnitude $F(r)$ where r is the distance from the origin. For such a particle,

(a) Write down the Lagrangian and set up the Euler-Lagrange equations.

(b) Show that the motion is planar

(c) Show that the differential equation governing the planar orbit is given by

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) = -\frac{\mu r^2}{l^2} F(r).$$

where r, θ refer to the plane polar coordinates, and l is the angular momentum of the particle with respect to the origin.

(d) Consider the motion of a particle under a force given by $F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3}$ where $k, \lambda > 0$. Obtain the general solution to the differential equation of the orbit.

[3] The electrostatic potential on a spherical surface $r = R$, centered at the origin, is given, in spherical polar coordinates (r, θ, ϕ), by

$$V(R, \theta, \phi) \;=\; V_0 \cos^2(\theta).$$

Assuming $V(r = \infty, \theta, \phi) = 0$,

(a) Find the potential everywhere.

(b) Find the electric field everywhere.

(c) What is the charge density on the spherical surface as a function of θ, ϕ?

(d) What is the total charge on the spherical surface?

[4.1] Using the Taylor expansion for a function $f(x)$, one can define functions $f(A)$ of a square matrix A. Thus consider the exponential of a matrix A through the Taylor expansion for e^x. Evaluate $\mathrm{Exp}[i\alpha\sigma^1]$ and reduce it to a simple expression, where α is a constant, $i^2 = -1$, and $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is one of the Pauli matrices.

[4.2] The wavefunction for a certain system satisfies the partial differential equation

$$\partial_r^2 \Psi + \frac{2}{r}\partial_r \Psi - (\partial_x^2 + \partial_y^2)\Psi = 0,$$

the ranges of the coordinates r, x, y being $[0, \infty],\ [-\infty, \infty],\ [-\infty, \infty]$ respectively.

(a) Assuming a separable form for the wavefunction, $\Psi(r, x, y) = e^{ik_x x + ik_y y}\phi(r)$, simplify the partial differential equation to obtain a second order ordinary differential equation for $\phi(r)$.

(b) Solve this ordinary differential equation and find the solution that is regular as $r \to 0$, and thereby the wavefunction. (*Hint:* Consider the substitution $\phi(r) = r^n f(r)$ for some real number n.)

[5.1] Suppose H is a time-independent hamiltonian, $\psi(t)$ a stationary state of H and A a time-independent observable. Calculate the rate of change of $< \psi(t)|A|\psi(t) >$ under Schrödinger evolution.

[5.2] Consider a particle in stationary state $\psi(t)$ subject to the non-relativistic hamiltonian $H = \frac{p^2}{2m} + V(x)$ in one-dimension. Calculate $\frac{d}{dt} < xp >$ in two different ways and thereby establish that $2 < T > \ = \ < xV'(x) >$. Here $< A >$ denotes expectation value of A in the stationary state $\psi(t)$.

[5.3] For a particle in a stationary state of the anharmonic oscillator, $H = \frac{p^2}{2m} + gx^4$, which is larger, the mean kinetic energy or the mean potential energy? Why?

[5.4] What is the probability amplitude to find the electron at the origin ($r = 0$) if it is in the energy eigenstate $n = 10,\ l = 5,\ m = 0$ of hydrogen. Justify your answer using Bohr's correspondence principle.

Useful Formulae

Laplacian in Spherical Polar Coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Expression for Legendre Polynomials:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$