CHENNAI MATHEMATICAL INSTITUTE

MSc Applications of Mathematics Entrance Examination

18 May 2015

Instructions:

- Enter your *Registration Number* here **CMI PG–** | | | | | | | | |
- Enter the name of the city where you write this test: | | | | | | | | | | | |
- The allowed time is 3 hours.
- This examination has two parts. Part A has multiple-choice questions, while questions in Part B require detailed answers.
- **The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.**
- **Answers to questions in Part A must be recorded on the sheet provided for the purpose.**
- You may use the blank pages at the end for your rough-work.

For office use only

Part B

Qno	1	2	3	4	5	6	7	8	9	10	11	12
Marks												

.	Part A	Part B	Total
Score			

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You must record your answers to Part A here by filling in the appropriate circles:
For example, if your answer to question number 7 is (A) and (D), record it as follows:

7. ● (B) (C) ●

Part A

1. (A) (B) (C) (D)

2. (A) (B) (C) (D)

3. (A) (B) (C) (D)

4. (A) (B) (C) (D)

5. (A) (B) (C) (D)

6. (A) (B) (C) (D)

7. (A) (B) (C) (D)

8. (A) (B) (C) (D)

9. (A) (B) (C) (D)

10. (A) (B) (C) (D)

Important

The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.

Part A

This section consists of Ten (10) multiple-choice questions, each with one or more correct answers. Record your answers on the attached sheet by filling in the appropriate circles.

Each question carries 5 marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

1. Let $p(x) = x^n + \sum_{k=0}^{n-1} a_k x^k$ and $q(x) = x^n + \sum_{k=0}^{n-1} b_k x^k$ be two polynomials with real coefficients such that $x = 3$ is a common root of the equations $p(x) = 0$ and $q(x) = 0$. Suppose $r(x)$ is the remainder when $p(x)$ is divided by the polynomial $q(x)$. Then we can conclude that

   (A) $r(3) = 0$.

   (B) $a_0 = b_0$.

   (C) $3a_1 + a_0 = 3b_1 + b_0$.

   (D) $r(x) = p(x) - q(x)$.

2. Let $p(x) = x^n + \sum_{k=0}^{n-1} a_k x^k$ and $q(x) = x^n + \sum_{k=0}^{n-1} b_k x^k$ be two polynomials with real coefficients such that $n \geq 4$ is even and $a_{n-1} < b_{n-1}$. Let $f(x)$ be a function such that $p(x) \leq f(x) \leq q(x)$ for all $x \in \mathsf{R}$. Then we can conclude that

   (A) $f(x)$ is a bounded function on R.

   (B) $f(x)$ is a continuous function on R.

   (C) There exists $x_0 \in \mathsf{R}$ such that $f(x_0) = 0$.

   (D) $f(x)$ is continuous at least at one point $x_0 \in \mathsf{R}$.

3. Let $f(x) = x^2 + \frac{1}{x^2}$ for $x \in (0, \infty)$. Then

   (A) f is a continuous function on $(0, \infty)$.

   (B) f is a uniformly continuous function on $(0, \infty)$.

   (C) f attains its infimum on $(0, \infty)$.

   (D) f attains its supremum on $(0, \infty)$.

4. Which of the following functions are continuous on R?

   (A) $f(x) = x\cos(x)$ for $x > 0$, $f(x) = -x\cos(x)$ for $x < 0$ and $f(0) = 0$.

   (B) $g(x) = \frac{\sin(x)}{x}$ for $x > 0$, $g(x) = \frac{-\sin(x)}{x}$ for $x < 0$ and $g(0) = 1$.

   (C) $h(x) = x$ for $x > 0$, $h(x) = -x$ for $x < 0$ and $h(0) = 0$.

   (D) $u(x) = e^x - 1$ for $x > 0$, $u(x) = 1 - e^{-x}$ for $x < 0$ and $u(0) = 0$.

5. In which of the following cases is the series $\sum_n a_n$ absolutely convergent?

   (A) $a_n = (-1)^n \frac{1}{n}$.
   (B) $a_n = (-1)^n \frac{(1-n^2)}{(1+n^4)}$.
   (C) $a_n = (1+(-1)^n 3)^{-n} n^2$.
   (D) $a_n = (1+(-1)^n 2)^{-n} n^2$.

6. Let a_n be a sequence of strictly positive numbers such that $\lim_{n\to\infty} \dfrac{a_n}{a_{n+1}} = 2$. Then

   (A) the radius of convergence of the series $\sum_n a_n x^n$ is $\frac{1}{2}$.
   (B) the radius of convergence of the series $\sum_n \frac{1}{a_n} x^n$ is $\frac{1}{2}$.
   (C) the radius of convergence of the series $\sum_n (a_n)^2 x^n$ is 4.
   (D) the radius of convergence of the series $\sum_n (a_n)^n x^n$ is ∞.

7. Which of the following functions is differentiable at $x = 0$? (Here $|a|$ denotes the absolute value of a real number a).

   (A) $f(x) = |x|(e^{-|x|} - 1)$.
   (B) $g(x) = x(e^{-|x|} - 1)$.
   (C) $h(x) = (e^{-|x|} - 1)$.
   (D) $u(x) = x|e^{-|x|} - 1|$.

8. Let $p(x)$ be an odd degree polynomial and let $q(x) = (p(x))^2 + 2p(x) - 2$.

   (A) The equation $q(x) = p(x)$ admits at least two distinct real solutions.
   (B) The equation $q(x) = 0$ admits at least two distinct real solutions.
   (C) The equation $p(x)q(x) = 4$ admits at least two distinct real solutions.
   (D) The equation $p(x) = 0$ admits at least two distinct real solutions.

9. Let $A = ((a_{ij}))$ be an $n \times n$- non-singular symmetric matrix such that each a_{ij} is a positive integer. Then we can conclude that

   (A) the determinant of A is a positive integer.
   (B) the trace of A is a positive integer.
   (C) the matrix A^{-1} has positive entries.
   (D) the matrix A^2 has positive entries.

10. Let $A = ((a_{ij}))$ be an $n \times n$- non-singular matrix such that each a_{ij} is a real number. Then we can conclude that (here I_n denotes the $n \times n$ identity matrix and A^t denotes the transpose of A)

   (A) The matix $I_n + A^2$ is a positive definite matrix.
   (B) The matix $I_n + AA^t$ is a positive definite matrix.
   (C) The matix $I_n + A$ is a positive definite matrix.
   (D) The matix $I_n + \frac{1}{2}(A + A^t)$ is a positive definite matrix.

Part B

Answer any five questions. Each question carries 10 marks. To get full credit, you must justify your answers.

1. Let Γ be a set with n elements and Λ be a set with m elements with $1 \leq n < m$. Find
   (a) the number of all mappings (functions) from Γ to Λ.
   (b) the number of all one-to-one mappings (injective functions) from Γ to Λ.
   (c) the number of all onto mappings (surjectve functions) from Γ to Λ.
   (d) the number of all one-to-one and onto mappings (bijective functions) from Γ to Λ.

2. Show that
$$1 + x \leq e^x \quad \forall x \in \mathsf{R}.$$

3. Let $a < b$ be real numbers. Show that the interval (a, b) contains a rational number as well as an irrational number.

4. Show that for all integers $k, r \geq 1$
$$\sum_{m=0}^{r} \binom{m+k-1}{k-1} = \binom{r+k}{k}.$$

5. Let $f : \mathsf{R} \mapsto \mathsf{R}$ be defined by
$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ x^4 & \text{if } x \text{ is irrational.} \end{cases}$$
   Is f differentiable at $x = 0$?

6. Let $f_n : [0, 1] \mapsto \mathsf{R}$ be a sequence of continous functions. Suppose that f_n converges uniformly to f. Show that f is continuous.

7. Let A, B be $n \times n$ matrices. Show that
$$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)).$$

8. Let A, B be $n \times n$ matrices and $\mathbf{c}$, $\mathbf{d}$ be $n \times 1$ vectors such that the matix equations
$$A\mathbf{x} = \mathbf{c}$$
$$B\mathbf{x} = \mathbf{d}$$
   are consistent, *i.e.*, each equation admits a solution. Can we conclude that
$$(A + B)\mathbf{x} = (\mathbf{c} + \mathbf{d})$$
   is also consistent? Prove if true or give a counter example if not true.