CHENNAI MATHEMATICAL INSTITUTE

MSc Applications of Mathematics Entrance Examination

15 May 2014

Instructions:

- Enter your *Registration Number* here **CMI PG–** ☐☐☐☐☐☐

  OR here **PG–** ☐☐☐☐
- Enter the name of the city where you write this test: ☐☐☐☐☐☐☐☐☐☐☐
- The allowed time is 3 hours.
- This examination has two parts. Part A has multiple-choice questions, while questions in Part B require detailed answers.
- **The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.**
- **Answers to questions in Part A must be recorded on the sheet provided for the purpose.**
- You may use the blank pages at the end for your rough-work.

For office use only

Part B

Qno	1	2	3	4	5	6	7	8	9	10	11	12
Marks												

.	Part A	Part B	Total
Score			

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You must record your answers to Part A here by filling in the appropriate circles:
For example, if your answer to question number 17 is (A) and (D), record it as follows:

17. ● (B) (C) ●

Part A

1. (A) (B) (C) (D)

2. (A) (B) (C) (D)

3. (A) (B) (C) (D)

4. (A) (B) (C) (D)

5. (A) (B) (C) (D)

6. (A) (B) (C) (D)

7. (A) (B) (C) (D)

8. (A) (B) (C) (D)

9. (A) (B) (C) (D)

10. (A) (B) (C) (D)

Important

The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.

Part A

This section consists of Ten (10) multiple-choice questions, each with one or more correct answers. Record your answers on the attached sheet by filling in the appropriate circles.

Every question is worth five (5) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

1. Suppose $\{a_n : n \geq 1\}$ is a sequence of positive numbers with $a_n \geq a_{n+1}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} a_n < \infty$. Then we can conclude that

   (a) $\lim_{n\to\infty} a_n = 0$.

   (b) $\lim_{n\to\infty} na_n = 0$.

   (c) $\lim_{n\to\infty} n^2 a_n = 0$.

   (d) $\lim_{n\to\infty} (\sum_{m=n}^{\infty} a_m) = 0$.

2. Which of the following functions is/are uniformly continuous on $(0, 1)$?

   (a) $f(x) = x(1 - x)$.

   (b) $g(x) = \frac{x}{(1-x)}$.

   (c) $u(x) = \frac{(1-x)}{x}$.

   (d) $v(x) = \frac{x}{(2-x)}$.

3. Let $f : (0, 1) \mapsto (0, 1)$ be a continuously differentiable function. Then we can conclude that

   (a) $g = \frac{1}{f}$ is a continuous function on $(0, 1)$.

   (b) $g = \frac{1}{f}$ is a continuously differentiable function on $(0, 1)$.

   (c) $g = \frac{1}{f}$ is a uniformly continuous function on $(0, 1)$.

   (d) h defined by $h(x) = x(1 - x)f(x)$ for $x \in (0, 1)$ is uniformly continuous.

4. Let A be a $n \times n$ non-singular symmetric matrix with entires in $(0, \infty)$. Then we can conclude that

   (a) $|A| > 0$ ($|A|$ denotes the determinant of A).

   (b) A is a positive definite matrix.

   (c) $B = A^2$ is a positive definite matrix.

   (d) $C = A^{-1}$ is a matrix with entires in $(0, \infty)$.

5. Let A be an $n \times n$ matrix with real entries and let $B = A^t A$ (where A^t is the transpose of A). Which of the following statements is/are always true?
   (a) If B is invertible then A must be invertible.
   (b) If λ is an eigenvalue of B then $\lambda \geq 0$.
   (c) If λ is an eigenvalue of A then λ^2 is an eigenvalue of B.
   (d) Let $C = I + B$. Then C is invertible.

6. Let A be a $m \times n$ matrix. Consider the equation

$$Ax = b \tag{1}$$

Which of the following is/are correct

(a) If the equation (1) admits a solution for all $b \in \mathsf{R}^m$ then n must be greater than or equal to m.

(b) If the equation (1) admits a unique solution for some $b \in \mathsf{R}^m$, then n must be greater than or equal to m.

(c) If the equation (1) admits two distinct solutions for some $b \in \mathsf{R}^m$, then m must be greater than or equal to n.

(d) If for all $b \in \mathsf{R}^m$, the equation (1) admits a solution that is unique then n must be equal to m.

7. Let f, g be real valued continuous functions on $[0, 1]$. Which of the following statements is/are always true?
(a) f is uniformly continuous.
(b) g is bounded.
(c) If $f(r) = 1 - g(r)$ for all rational numbers $r \in [0, 1]$ then $f(x) = 1 - g(x)$ for all $x \in [0, 1]$.
(d) The function v defined by $v(x) = f^2(x) + g^2(x)$, $0 \leq x \leq 1$ is continuous.

8. Let A be an $n \times m$ matrix and B be an $m \times n$ matrix. For a square matrix D, let $Tr(D)$ denote trace of D, $|D|$ denote the determinant of D. Suppose that AB is invertible. Then which of the following is/are always true?
(a) $Tr(AB) = Tr(BA)$.
(b) $|AB| = |BA|$.
(c) $m \geq n$.
(d) BA must be invertible.

9. Let f be a continuously differentiable function on R (and let f' denote the derivative of f). Which of the following statements is/are always true?
(a) f is uniformly continuous on $(-1, 1)$.
(b) If $f'(x) \neq 0$ for all $x \in (a, b)$ then

$$\max(f(a), f(b)) > f(x) \ \ \forall x \in (a, b).$$

(c) If f' is a bounded function, then f is a uniformly continuous function on R.
(d) If f is a bounded function, then f' is a uniformly continuous function on R.

10. Let $\{a_n : n \geq 1\}$ be a sequence of nonnegative numbers such that

$$\lim_{n \to \infty} (a_n)^{\frac{1}{n}} = 1.$$

Which of the following statements is/are always true?
(a) The series $\sum_n a_n$ converges.
(b) The series $\sum_n a_n x^n$ converges uniformly for $x \in [-0.5, 0.5]$.
(c) The series $\sum_n a_n x^n$ converges uniformly for $x \in [-1, 1]$.
(d) $\limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$.

Part B

Answer any six questions. Each question carries 10 marks.

1. In how many ways can 7 girls and 8 boys be seated in a line so that no two girls are together.

2. Is the series given below convergent? Justify.

$$\sum_{2}^{\infty} \frac{\log(n+1) - \log n}{\sqrt{n}}.$$

3. Let $t > 0$ be fixed. Show that

$$\frac{1}{n}\left\{\sin\frac{t}{n} + \sin\frac{2t}{n} + \sin\frac{3t}{n} + \cdots + \sin\frac{(n-3)t}{n}\right\} \to \frac{1-\cos t}{t}.$$

4. Show that

$$\int_0^{2\pi n} \frac{\sin(n^3 x)}{n^2 + x^2}\, dx \to 0.$$

5. Does the series $\sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{n^2}$ converge uniformly for $x \in \mathbb{R}$.

6. Let f be a continuous function on an open interval (a, b) and let $x_1, x_2, \cdots, x_n \in (a, b)$. Show that there exists a number $u \in (a, b)$ such that

$$f(u) = \frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

7. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every value exactly twice *i.e.* (here $\#(A)$ denotes number of elements in the set A).

$$\#\{x \in \mathsf{R} : f(x) = y\} = 2 \quad \forall y \in \mathsf{R}.$$

8. Let

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

   Show that f is continuous function on R^2.

9. The numbers 2475; 1287; 4598; 5742 are all divisible by 11. Show that the following determinant is also divisible by 11.

$$\begin{vmatrix} 2 & 1 & 4 & 5 \\ 4 & 2 & 5 & 7 \\ 7 & 8 & 9 & 4 \\ 5 & 7 & 8 & 2 \end{vmatrix}.$$

10. Let T be the linear transform of R^3 given by

$$T(1, 0, 0) = (1, 0, 0); \quad T(1, 1, 0) = (1, 1, 1); \quad T(1, 1, 1) = (1, 1, 0).$$

   find formula for $T(x, y, z)$. Find Kernel(T). Find range(T). Show $T^3 = T$.

11. Let V be the linear space of functions on R spanned by the functions $\{1, x, e^x, xe^x\}$. Define linear map $T : V \to V$ by $T(f) = f'$, the derivative of f. Calculate the matrix of T w.r.t. the above basis.

12. Let $T : R^3 \to R^3$ be a linear transformation such that $T \not\equiv 0$ but $T^2 \equiv 0$. Show that

$$\dim(\text{Range}(T)) = 1.$$