CHENNAI Mathematical Institute

MSc Applications of Mathematics

Entrance Examination, 2012

Part A

Answer any 6 questions. Each question carries 10 marks. For each of the statements given below, state whether it is True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided.

1. Let $\{a_n : n \geq 1\}$ be a sequence of real numbers such that the radius of convergence R of the power series $p(t) = \sum_{m=0}^{\infty} a_n t^n$ satisfies $R > 0$. Then a_n converges to 0.

2. Let $\mathbf{A}$ be a symmetric $n \times n$ matrix and suppose that A is positive definite. Then
$$a_{jk} \leq \frac{1}{2}(a_{jj} + a_{kk}).$$

3. Let $p(x)$ be a polynomial and $a > 0$ be such that $p(a) > 0$. Let $q(x) = p(x) - p(a)$. Then $(x - a)$ is a factor of $q(x)$ but $(x - a)^2$ is not a factor of $q(x)$.

4. Let $f(x) = |x| \sin(x)$ and $g(x) = |x| \cos(x)$ for $-2\pi \leq x \leq 2\pi$. Then f and g are differentiable on $[-2\pi, 2\pi]$.

5. Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be $n \times n$ matrices that are positive definite such that
$$a_{ij} < b_{ij} \ \ \forall i, j.$$
Let $\mathbf{C} = (c_{ij})$ be defined by $c_{ij} = a_{ij} - b_{ij}$. Then $\mathbf{C}$ is also positive definite.

6. Let f_n be a sequence of continuous functions on $[0, 1]$ converging to f point wise, where f is a continuous function. Then f_n converges uniformly to f.

7. Let f be a continuous function on $(0, 1)$ taking values in $[0, 1]$ such that
$$f(x) \leq x(1 - x) \quad \forall x \in (0, 1).$$
Then f is uniformly continuous on $(0, 1)$.

8. Let g be defined by
$$g(x) = |x|^3 \exp\{-|x|\} \ \ x \in \mathbb{R}.$$
Then g is a continuously differentiable function on $\mathbb{R}$.

9. Suppose λ^2 is an eigenvalue of $\mathbf{A}^2$. Then λ is an eigenvalue of $\mathbf{A}$.

Part B

Answer any four questions. Each question carries 10 marks.State precisely any theorem that you use.

1. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ be such that
$$\|\mathbf{x} + t\mathbf{y}\| \geq \|x\|, \quad \forall t \in \mathbb{R}.$$
Show that $\mathbf{x} \cdot \mathbf{y} = 0$.

2. Let $p(x)$ be a n^{th} degree polynomial such that the equation $p(x) = 0$ admits n distinct real roots $c_1, c_2, \ldots c_n$. Suppose that $p(x) \neq 0$ for $-1 < x < 1$. Show that $|c_j| \leq |p(0)|$ for $j = 1, 2, \ldots n$.

3. Let $f(x) = |x| \exp\{-x\}$ for $-1 \leq x \leq 1$. Find $u, v \in [-1, 1]$ such that
$$f(u) \leq f(x) \leq f(v), \quad \forall x \in [-1, 1].$$

4. Let $\mathbf{A}$ be a $n \times n$ matrix and $\mathbf{y}$ be a $n \times 1$ matrix (vector) such that the equation
$$\mathbf{A}\mathbf{x} = \mathbf{y}$$
for a $n \times 1$ matrix (vector) $\mathbf{y}$ admits no solution. Show that the rank of $\mathbf{A}$ is strictly less than n.

5. Let $\mathbf{A} = (a_{ij})$ be a 100×100 matrix defined by
$$a_{ij} = i^2 + j^2.$$
Find the rank of $\mathbf{A}$.

6. For $n \geq 1$ let
$$a_n = \frac{(\log n)^4}{n^2}.$$
Show that the series
$$\sum_{n=1}^{\infty} a_n$$
converges.