CHENNAI Mathematical Institute

MSc Applications of Mathematics

Entrance Examination, 2011

Part A

For each of the statements given below, state whether it is True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided.

1. $(k!)^2$ is a factor of $(2k+2)!$ for any positive integer k.
2. Let $p(x)$ be a polynomial of degree 101. Then $x \mapsto p(x)$ can not be a one-one onto mapping (i.e. bijective function) from $\mathbb{R}$ to $\mathbb{R}$.
3. For all $x > 0$
$$e^{-x} \geq 1 - x.$$
4. For all even integers $n = 2m > 1$,
$$\sum_{k=1}^{m} 2^{2k-1} \binom{n}{2k-1} = \sum_{k=1}^{m} 2^{2k} \binom{n}{2k}$$
5. Let $f(x) = |x|^3(1+x^2)$ for $x \in \mathbb{R}$. Then f is not differentiable at $x = 0$ and is differentiable at all other points.
6. Let $\{a_n\}$ be a sequence of strictly positive real numbers such that
$$\lim_{n\to\infty} a_n = 0.$$
Let $c_n = na_n$ and $d_n = (a_n)^2$. Then the sequences $\{c_n\}, \{d_n\}$ also converge to zero.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x^2}{1+x^2}$, $x \in \mathbb{R}$. Then f attains its supremum.
8. Let $f : [1, \infty) \mapsto \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Then f is uniformly continuous.
9. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$- matrices each having rank n. Then rank of $\mathbf{C} = \mathbf{A^3B^2A}$ is also n.
10. Let n be greater than 5 and $\mathbf{A}$ be a $n \times n$-matrix and $\mathbf{b}$ is a $n \times 1$ vector such that the equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution. Then the rank of $\mathbf{A}$ is strictly less than n.

Part B

Answer any five questions. State precisely any theorem that you use.

1. Let f be a continuously differentiable function on $[0,1]$. Show that there exists a number M such that for every partition $0 = x_0 < x_1 < \ldots < x_k = 1$ of $[0,1]$ (for any k), one has
$$\sum_{j=1}^{k} |f(x_j) - f(x_{j-1})| \leq M.$$

2. For $0 < x < \infty$, let $f(x) = \sum_{n=0}^{\infty} \exp\{-nx\}$. Show that $f(x)$ is a continuous function.

3. Let $A = (a_{ij})$ be a $n \times n$ matrix where $a_{ij} = \min(i,j),\ 1 \leq i,j \leq n$. Find the rank of A.

4. Let $f(x) = (\cos(x))^7 + (\sin(x/2))^2 + 1$ for $0 \leq x \leq \pi$. Calculate
$$\int_0^{\pi} f(x)dx.$$

5. Let f be a continuously differentiable function on $[0,\infty)$ such that $f'(x) \leq f(x)$ for all $x \in [0,\infty)$, where $f'(x)$ denotes the derivative of f at x. Suppose $f(0) = 5$. Show that $f(x) \leq 5\exp\{x\}$ for all $x \in [0,\infty)$.

6. Give an example of sequences $\{a_n\}$ and $\{b_n\}$ such that both sequences converge to zero as n tends to infinity and the sequence $c_n = \frac{(a_n)^2}{b_n}$ converges to 5.

7. Suppose that the power series $p(x) = \sum b_n x^n$ converges for $|x| \leq 1$. Suppose that for some $\delta > 0$, $p(x) = 0$ for $|x| < \delta$. Show that $b_n = 0$ for all $n \geq 1$.

8. Suppose 5 boys and 3 girls are to be arranged in a queue such that only one girl has a boy adjacent to her. Find the number of such arrangements possible.

9. Let $\mathbf{A}$ be a $n \times n$ matrix with real entries. Show that there exist $n \times n$ matrices B, C with real entries such that $A = B + C$ and B is symmetric and C is anti-symmetric (*i.e.* $B^T = B$ and $C^T = -C$, B^T denoting the transpose of B).

10. Give an example of a 2×2 matrix A with real entries such that $A^2 = -4I$ where I is the identity matrix.