2013 Entrance Examination for BSc Programmes at CMI

Part A. (10 problems $\times$ 5 points = 50 points.) **Attempt all questions in this part before going to part B. Carefully read the details of marking scheme given below. Note that wrong answers will get negative marks!**

In each problem you have to fill in 4 blanks as directed. Points will be given based only on the filled answer, so you need not explain your answer. Each correct answer gets 1 point and having all 4 answers correct will get 1 extra point for a total of 5 points per problem. But each wrong/illegible/unclear answer will get minus 1 point. Negative points from any problem will be counted in your total score, so it is better not to guess! If you are unsure about a part, you may leave it blank without any penalty. If you write something and then want it not to count, cross it out and clearly write "no attempt" next to the relevant part.

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1. For sets A and B, let $f : A \to B$ and $g : B \to A$ be functions such that $f(g(x)) = x$ for each x. For each statement below, write whether it is TRUE or FALSE.

a) The function f must be one-to-one.

Answer: ________

b) The function f must be onto.

Answer: ________

c) The function g must be one-to-one.

Answer: ________

d) The function g must be onto.

Answer: ________

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, where $\mathbb{R}$ is the set of real numbers. For each statement below, write whether it is TRUE or FALSE.

a) If $|f(x) - f(y)| \leq 39|x - y|$ for all x, y then f must be continuous everywhere.

Answer: ________

b) If $|f(x) - f(y)| \leq 39|x - y|$ for all x, y then f must be differentiable everywhere.

Answer: ________

c) If $|f(x) - f(y)| \leq 39|x - y|^2$ for all x, y then f must be differentiable everywhere.

Answer: ________

d) If $|f(x) - f(y)| \leq 39|x - y|^2$ for all x, y then f must be constant.

Answer: ________

3. Let S be a circle with center O. Suppose A, B are points on the circumference of S with $\angle AOB = 120°$. For triangle AOB, let C be its circumcenter and D its orthocenter (i.e., the point of intersection of the three lines containing the altitudes). For each statement below, write whether it is TRUE or FALSE.

a) The triangle AOC is equilateral.

Answer: ________

b) The triangle ABD is equilateral.

Answer: ________

c) The point C lies on the circle S.

Answer: ________

d) The point D lies on the circle S.

Answer: ________

4. A polynomial $f(x)$ with real coefficients is said to be a sum of squares if we can write $f(x) = p_1(x)^2 + \cdots + p_k(x)^2$, where $p_1(x), \ldots, p_k(x)$ are polynomials with real coefficients. For each statement below, write whether it is TRUE or FALSE.

a) If a polynomial $f(x)$ is a sum of squares, then the coefficient of every odd power of x in $f(x)$ must be 0.

Answer: ________

b) If $f(x) = x^2 + px + q$ has a non-real root, then $f(x)$ is a sum of squares.

Answer: ________

c) If $f(x) = x^3 + px^2 + qx + r$ has a non-real root, then $f(x)$ is a sum of squares.

Answer: ________

d) If a polynomial $f(x) > 0$ for all real values of x, then $f(x)$ is a sum of squares.

Answer: ________

5. There are 8 boys and 7 girls in a group. For each of the tasks specified below, write an expression for the number of ways of doing it. Do NOT try to simplify your answers.

a) Sitting in a row so that all boys sit contiguously and all girls sit contiguously, i.e., no girl sits between any two boys and no boy sits between any two girls

Answer:

b) Sitting in a row so that between any two boys there is a girl and between any two girls there is a boy

Answer:

c) Choosing a team of six people from the group

Answer:

d) Choosing a team of six people consisting of *unequal* number of boys and girls

Answer:

6. Calculate the following integrals whenever possible. If a given integral does not exist, state so. Note that $[x]$ denotes the integer part of x, i.e., the unique integer n such that $n \leq x < n+1$.

a) $\int_1^4 x^2 dx$

Answer: ________

b) $\int_1^3 [x]^2 dx$

Answer: ________

c) $\int_1^2 [x^2] dx$

Answer: ________

d) $\int_{-1}^1 \frac{1}{x^2} dx$

Answer: ________

7. Let A, B, C be angles such that e^{iA}, e^{iB}, e^{iC} form an equilateral triangle in the complex plane. Find values of the given expressions.

a) $e^{iA} + e^{iB} + e^{iC}$

Answer: ________

b) $\cos A + \cos B + \cos C$

Answer: ________

c) $\cos 2A + \cos 2B + \cos 2C$

Answer: ________

d) $\cos^2 A + \cos^2 B + \cos^2 C$

Answer: ________

8. Consider the quadratic equation $x^2 + bx + c = 0$, where b and c are chosen randomly from the interval [0,1] with the probability uniformly distributed over all pairs (b, c). Let $p(b)$ = the probability that the given equation has a real solution for *given* (fixed) value of b. Answer the following questions by filling in the blanks.

a) The equation $x^2 + bx + c = 0$ has a real solution if and only if $b^2 - 4c$ is

Answer: ________

b) The value of $p(\frac{1}{2})$, i.e., the probability that $x^2 + \frac{x}{2} + c = 0$ has a real solution is

Answer: ________

c) As a function of b, is $p(b)$ increasing, decreasing or constant?

Answer: ________

d) As b and c both vary, what is the probability that $x^2 + bx + c = 0$ has a real solution?

Answer: ________

9. Let $\mathbb{R}$ = the set of real numbers. A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(1) = 1$, $f(2) = 4$, $f(3) = 9$ and $f(4) = 16$. Answer the independent questions below by choosing the correct option from the given ones.

a) Which of the following values must be in the range of f ?
Options: 5 25 both neither

Answer: ________

b) Suppose f is differentiable. Then which of the follwing intervals must contain an x such that $f'(x) = 2x$? **Options:** (1,2) (2,4) both neither

Answer: ________

c) Suppose f is twice differentiable. Which of the following intervals must contain an x such that $f''(x) = 2$? **Options:** (1,2) (2,4) both neither

Answer: ________

d) Suppose f is a polynomial, then which of the following are possible values of its degree?
Options: 3 4 both neither

Answer: ________

10. Let

$$f(x) = \frac{x^4}{(x-1)(x-2)\cdots(x-n)}$$

where the denominator is a product of n factors, n being a positive integer. It is also given that the X-axis is a horizontal asymptote for the graph of f. Answer the independent questions below by choosing the correct option from the given ones.

a) How many vertical asymptotes does the graph of f have?
Options: n less than n more than n impossible to decide

Answer: ________

b) What can you deduce about the value of n ?
Options: $n < 4$ $n = 4$ $n > 4$ impossible to decide

Answer: ________

c) As one travels along the graph of f from left to right, at which of the following points is the sign of $f(x)$ guaranteed to change from positive to negative?
Options: $x = 0$ $x = 1$ $x = n - 1$ $x = n$

Answer: ________

d) How many inflection points does the graph of f have in the region $x < 0$?
Options: none   1   more than 1   impossible to decide
(Hint: Sketching is better than calculating.)

Answer: ________

Part B. (Problems 1–4 $\times$ 15 points + problems 5–6 $\times$ 20 points = 100 points.) Solve these problems in the space provided for each problem after this page. You may solve only part of a problem and get partial credit. **Clearly explain your entire reasoning. No credit will be given without reasoning.**

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1. In triangle ABC, the bisector of angle A meets side BC in point D and the bisector of angle B meets side AC in point E. Given that DE is parallel to AB, show that AE = BD and that the triangle ABC is isosceles.

2. A curve C has the property that the slope of the tangent at any given point (x, y) on C is $\frac{x^2+y^2}{2xy}$.

a) Find the general equation for such a curve. Possible hint: let $z = \frac{y}{x}$.

b) Specify all possible shapes of the curves in this family. (For example, does the family include an ellipse?)

3. A positive integer N has its first, third and fifth digits equal and its second, fourth and sixth digits equal. In other words, when written in the usual decimal system it has the form $xyxyxy$, where x and y are the digits. Show that N cannot be a perfect power, i.e., N cannot equal a^b, where a and b are positive integers with $b > 1$.

4. Suppose $f(x)$ is a function from $\mathbb{R}$ to $\mathbb{R}$ such that $f(f(x)) = f(x)^{2013}$. Show that there are infinitely many such functions, of which exactly four are polynomials. (Here $\mathbb{R}$ = the set of real numbers.)

5. Consider the function $f(x) = ax + \frac{1}{x+1}$, where a is a *positive* constant. Let L = the largest value of $f(x)$ and S = the smallest value of $f(x)$ for $x \in [0, 1]$. Show that $L - S > \frac{1}{12}$ for any $a > 0$.

6. Define $f_k(n)$ to be the sum of all possible products of k distinct integers chosen from the set $\{1, 2, \ldots, n\}$, i.e.,

$$f_k(n) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} i_1 i_2 \ldots i_k \ .$$

a) For $k > 1$, write a recursive formula for the function f_k, i.e., a formula for $f_k(n)$ in terms of $f_\ell(m)$, where $\ell < k$ or ($\ell = k$ and $m < n$).

b) Show that $f_k(n)$, as a function of n, is a polynomial of degree $2k$.

c) Express $f_2(n)$ as a polynomial in variable n.