# INSTITUTE OF ACTUARIES OF INDIA

# EXAMINATIONS

## 28th April 2016

## Subject CT4 – Models

## Time allowed: Three Hours (10.30 – 13.30 Hrs)

## Total Marks: 100

### INSTRUCTIONS TO THE CANDIDATES

1. ***Please read the instructions on the front page of answer booklet and instructions to examinees sent along with hall ticket carefully and follow without exception***

2. ***Mark allocations are shown in brackets.***

3. ***Attempt all questions, beginning your answer to each question on a separate sheet.***

4. ***Please check if you have received complete Question Paper and no page is missing. If so, kindly get new set of Question Paper from the Invigilator.***

**AT THE END OF THE EXAMINATION**

**Please return your answer book and this question paper to the supervisor separately.**

**Q. 1)** A Reinsurance company, where you work, currently wants to create an independent model for catastrophe pricing. Your Manager has asked you work on this. List the advantages and disadvantages of using models for pricing a risk. **[4]**

**Q. 2)** In the context of a stochastic process {Xt : t, J}, explain the meaning of the following conditions:

**i)** Strict Stationary (1.5)

**ii)** Weak Stationary (1.5)

**iii)** Mixed Process (1.5)

**iv)** Counting Process (1.5)
**[6]**

**Q. 3)** **i)** Two groups of 15 leukemia patients were given different treatments. The time to remission (relapse) in weeks for the two groups is given below. (+ indicates that the observation is censored)

Group A: 5, 3+, 6, 6, 7, 10, 10, + 6+, 1, 1, 2, 4, 4, 5, 9+

Group B: 3, 3, 2, 2, 1, 1, 1, 3, 4, 2, 1, 4, 5, 6, 5

Plot the Kaplan-Meir survival function for the 2 groups and comment on your observations. (5)

**ii)** Estimate the median for the 2 groups (1)
**[6]**

**Q. 4)** Claims arrive at an insurance company according to a Poisson Process with rate λ per year.

**i)** Show that, given that there is exactly one claim in the time interval [t, t + s], the time of the claim arrival is uniformly distributed on [t, t + s]. (3)

**ii)** State the joint density of the holding times $T_0, T_{2 \ldots\ldots\ldots}, T_n$ between successive claims. (1)

**iii)** Show that, given that there are n claims in the time interval [0, t], the number of claims in the interval [0, s] for s < t is binomial with parameters n and s/t. (3)
**[7]**

**Q. 5)** **i)** Define a proportional hazards model and give the formula used for such models, explaining what the baseline hazard denotes (2)

**ii)** Explain the difference between the parametric and semi-parametric approaches for proportional hazards model. (2)

The survival time after a heart surgery is modeled using a Cox proportional hazards model with smoking-status, body mass index, exercise-score (ranging from 1 to 4 for 30 minutes of exercise to 2 hours of exercise. The co-variates for $i$th life are $z_i = (X_{i1}, X_{i2}, X_{i3})$ where

$X_1$ = 1 for smoker and 0 for non-smoker
$X_2$ = body mass index
$X_3$ = exercise score

The regression parameters are $\beta = (\beta_1, \beta_2, \beta_3)$

The following model has been suggested for the force of mortality:

$$\mu(x) = \mu_0(x) * e^{(\beta 1+\beta 2+\beta 3)}$$
$$\mu(x) = \mu_0(x) * e^{(\beta z)}$$

A group of 10 persons aged exactly 55 years who had just undergone a heart surgery were observed for a period of 10 years.

| Life | Age at exit | Smoking-status | BMI | Exercise-score | Reason for exit |
|---|---|---|---|---|---|
| 1 | 57 | 1 | 25 | 2 | Censored |
| 2 | 65 | 0 | 25 | 4 | Survived |
| 3 | 58 | 0 | 30 | 1 | Died |
| 4 | 59 | 1 | 35 | 2 | Censored |
| 5 | 60 | 0 | 35 | 1 | Died |
| 6 | 65 | 0 | 30 | 3 | Survived |
| 7 | 56 | 1 | 35 | 1 | Died |
| 8 | 62 | 1 | 30 | 3 | Censored |
| 9 | 63 | 0 | 30 | 2 | Died |
| 10 | 60 | 0 | 30 | 1 | Censored |

**iii)** Give the partial likelihood for these observations, stating any assumptions made (5)

**iv)** Explain, how based on the above observations, would an estimates of the parameters be obtained (1)

You have estimated the parameters to be

$\beta_1$= 0.05, $\beta_2$= 0.01, $\beta_3$= - 0.1,

**v)** Estimate what the exercise score should be for a non-smoker aged 64 with body mass index of 35 to have the same risk of dying as another non-smoker aged 64 with body mass index of 25 and exercise score of 3. (2)

**[12]**

**Q. 6)** The number of daily-deaths in a hospital has been observed over a period of five years and the observations are given below.

| | Calendar Year | | | | |
|---|---|---|---|---|---|
| **No of deaths** | **2011** | **2012** | **2013** | **2014** | **2015** |
| 0 | 239 | 229 | 222 | 224 | 220 |
| 1 | 100 | 106 | 115 | 106 | 113 |
| 2 | 24 | 27 | 22 | 30 | 23 |
| 3 | 2 | 4 | 4 | 5 | 8 |
| 4 | 0 | 0 | 2 | 0 | 1 |

**i)** Assuming that the number of daily-deaths follows a Poisson distribution and that the distribution is same across all years, estimate the parameter of the Poisson distribution. (1.5)

**ii)** Check for the goodness of fit of the fitted model and comment on your observations. (2.5)

**iii)** Calculate the probability of observing 6 deaths in a day. (1)

A similar distribution of daily-deaths in another hospital over the same period is given below.

| | Calendar Year | | | | |
|---|---|---|---|---|---|
| **No of deaths** | **2011** | **2012** | **2013** | **2014** | **2015** |
| 0 | 97 | 91 | 93 | 67 | 63 |
| 1 | 115 | 120 | 115 | 119 | 118 |
| 2 | 98 | 89 | 88 | 94 | 95 |
| 3 | 35 | 42 | 45 | 54 | 55 |
| 4 | 16 | 19 | 17 | 21 | 24 |
| 5 | 4 | 3 | 6 | 8 | 9 |
| 6 | 0 | 2 | 1 | 2 | 1 |

**iv)** Assuming that the number of daily-deaths in this hospital also follows a Poisson distribution and that the distribution is same across all years, estimate the parameter of the Poisson distribution. (1)

**v)** Check for the goodness of fit for the deaths in year 2014. (2.5)

**vi)** Based on your observations, suggest any alternative approach for estimating the Poisson parameter and check for the goodness of fit of the alternative approach for the deaths in 2014. (3.5)

**[12]**

**Q. 7)** **i)** A life office is conducting a study into the mortality of female lives ages 35, within 1 year of child-birth. The age is defined as age last birthday and the number of deaths and exposed to risk calculated accordingly. The below gives an extract of 5 lives. Calculate the exposure of each of the life for the above mortality analysis.

| | Date of birth | Date of child birth | Remarks |
|---|---|---|---|
| **A** | 1.1.1980 | 1.11.2014 | Died on 1st May 2015 |
| **B** | 1.2.1980 | 1.2.2015 | Policy lapsed on 31st Jan 2015 |
| **C** | 1.4.1980 | 1.12.2014 | Died on 1st July 2015 |
| **D** | 1.10.1981 | 1.1.2015 | - |
| **E** | 1.5.1979 | 1.1.2014 | - |

(2)

**ii)** **a)** State the rate interval implied by the age definition above. (1)

**b)** If age were to be defined as age last birthday on the preceding 1 April, explain the difference in the rate interval implied by this definition and the definition in part ii (a). (1)

**iii)** The mortality investigation of a Government pension scheme is being done. The data has only the year of birth and year of death. Hence age at death is defined as Year of death – Year of birth.

**a)** State the rate year implied by this definition and give the age-range at the start of the rate year. (2)

You have been provided with the following data

- For all deaths in year 2013, the year of birth and year of death
- The data as at 1st July 2012, 1st July 2013 and 1st July 2014 with member wise details of year of birth

**b)** Explain how you would use the above data to derive the mortality rates of the pensioners (3)

**[9]**

**Q. 8)** A time-inhomogeneous Markov jump process has two state space {Funded, Unfunded} and the transition rate for switching between states equals $2t$, regardless of the state currently occupied, where $t$ is time.

The process starts in state "Funded" at $t = 0$.

**i)** Calculate the probability that the process remains in state Funded until at least time s. (3)

**ii)** Calculate the probability that the process is in state Unfunded at time T, and that it is in the first visit to state Unfunded. (4)

**iii)** **a)** Sketch the probability function given in (ii). (2)

**b)** Give an explanation of the shape of the probability function. (2)

**c)** Calculate the time at which it is most likely that the process is in its first visit to state Unfunded. (2)

**[13]**

**Q. 9)** **i)** Explain the flaws in chi-squared test used for testing the effectiveness of graduation (2)

**ii)** Describe two alternative tests that can be used to detect the above flaws, explaining how these tests remove the defects in chi-squared test (2)

The mortality experience for a group of lives was observed and the graduated mortality rates were estimated using the equation $q_{x\,=\,\text{logit}-1\,(a_x)}$the parameter '$a$' being estimated from the observations.

The following table gives the number of individuals and deaths observed for 20 different ages, along with the individual standardized deviations of the observed deaths from the graduated deaths for each age.

| Age | Number of lives | Deaths observed | Std deviation, Zx |
|---|---|---|---|
| 15 | 9120 | 18 | -1.1248 |
| 18 | 8703 | 20 | 0.1992 |
| 25 | 7942 | 25 | 0.8671 |
| 27 | 7750 | 28 | 0.6222 |
| 31 | 7365 | 34 | 2.3095 |
| 35 | 6992 | 38 | 1.5452 |
| 38 | 6525 | 40 | 2.5190 |
| 42 | 5876 | 39 | -0.3668 |
| 46 | 5574 | 42 | -0.5783 |
| 49 | 5132 | 47 | -0.5772 |
| 52 | 4823 | 50 | -2.1147 |
| 57 | 4537 | 55 | 1.0562 |
| 63 | 4238 | 57 | 0.5871 |
| 67 | 3650 | 60 | -0.4615 |
| 71 | 3112 | 65 | -0.3813 |
| 75 | 2701 | 69 | -2.3091 |
| 77 | 2233 | 70 | -0.5264 |
| 82 | 1657 | 74 | -0.3860 |
| 85 | 845 | 78 | -0.0999 |
| 87 | 412 | 80 | -0.5035 |

The chi-squared statistic is 29.6253 which is lower than 30.144 ($\aleph^2$ critical value with 19 degrees of freedom) and hence there is no evidence to reject the graduated rates.

**iii)** Use the two alternate tests described above, to validate the observation under chi-squared test. (5)

**iv)** In a life insurance company the mortality investigation is conducted based on policies, rather than lives. Lives aged exactly 45 is observed for a year, from age 45 to 46. Given that one life can have more than 1 policy, prove that the expected mean of deaths is same whether mortality is observed by lives or policies. Assume that the mortality rate for age 45 is 0.005. (2)

The following table gives the distribution of policies taken by insured lives aged exactly 45. These lives are observed for a year, from age 45 to 46.

| No of policies | Individuals |
|---|---|
| 1 | 1257 |
| 2 | 105 |
| 3 | 40 |
| 4 | 35 |
| 5 | 23 |
| 6 | 16 |
| 7 | 11 |
| 8 | 7 |
| 9 | 4 |
| 10 | 2 |
| **Total** | **1500** |

**v)** Calculate the variance ratio for the above population. (2)

**[13]**

**Q. 10)** An accidental Car arriving at the Garage to repair (A state) waits for an average of one hour before being classified by a Junior Mechanic as requiring Major work (State M), Minor work (State S) or further investigation (State F). Only one new arrival in ten is classified as Major work, five in ten as Minor Work.

If needed, further investigation by senior mechanic takes an average of 3 hours, after which 50% cases are not required any work (State O), 25% are sent to receive Minor work and 25% sent for major work category.

Minor work category takes an average of 2 hours to complete and major work category takes an average of 60 hours of repair and both result transfer in to state O. It is suggested that a time-homogeneous Markov process with states A, F, M, S and O could be used to model the progress of Car repairing through the system, with the ultimate aim of reducing the average time spent in the Garage.

**i)** Write down the matrix of transition rates, $\{\sigma_{ij} : i, j = A,F,M,S,O\}$, of above model. (3)

**ii)** Calculate the proportion of Car who eventually receives Major work of repair. (1)

**iii)** Derive expressions for the probability that a Car arriving at time 0 is:

**a)** Yet to be classified by the Junior Mechanic at time t, and (1.5)

**b)** Undergoing further investigation at time t (1.5)

**iv)** Let $m_i$ denote the expectation of the time until transferring in State 'O' for transfer of Car currently in State i.

**a)** Explain in words why $m_i$ satisfies the following equation:

$$m_i = 1/\lambda_i + \sum_{j \neq i,k} \left( \frac{\sigma_{ij}}{\lambda i} * m_j \right)$$

Where $\lambda_i = \sum \sigma ij$ (2)

**b)** Hence calculate the expectation of the total time until transfer in to State 'O' for a Car arrived in to garage. (2)

**v)** State the distribution of the time spent in each State visited according to this model (1)

The average times listed above may be assumed to be the sample means waiting times derived from tracking a large sample of cars through the system.

**vi)** Describe briefly what additional feature of the data might be used to check that this simple model matches the situation being modeled. (2)

**vii)** The Garage owner believes that replacing the Junior Mechanic with a more Senior Mechanic will save resources by reducing the proportion of cases sent for further investigation i.e. State 'F'. Alternatively, the same resources could go towards reducing minor issue resolving time.

**a)** Outline briefly the calculations that would need to be performed to compare the options. (2)

**b)** Discuss whether the current model is suitable as a basis for making decisions of this nature. (2)

**[18]**

******************************