

# 29<sup>th</sup> Indian National Mathematical Olympiad-2014

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1. In a triangle  $ABC$ , let  $D$  be a point on the segment  $BC$  such that  $AB + BD = AC + CD$ . Suppose that the points  $B, C$  and the centroids of triangles  $ABD$  and  $ACD$  lie on a circle. Prove that  $AB = AC$ .

**Solution.** Let  $G_1, G_2$  denote the centroids of triangles  $ABD$  and  $ACD$ . Then  $G_1, G_2$  lie on the line parallel to  $BC$  that passes through the centroid of triangle  $ABC$ . Therefore  $BG_1G_2C$  is an isosceles trapezoid. Therefore it follows that  $BG_1 = CG_2$ . This proves that  $AB^2 + BD^2 = AC^2 + CD^2$ . Hence it follows that  $AB \cdot BD = AC \cdot CD$ . Therefore the sets  $\{AB, BD\}$  and  $\{AC, CD\}$  are the same (since they are both equal to the set of roots of the same polynomial). Note that if  $AB = CD$  then  $AC = BD$  and then  $AB + AC = BC$ , a contradiction. Therefore it follows that  $AB = AC$ .

2. Let  $n$  be a natural number. Prove that

$$\left[ \frac{n}{1} \right] + \left[ \frac{n}{2} \right] + \left[ \frac{n}{3} \right] + \cdots + \left[ \frac{n}{n} \right] + [\sqrt{n}]$$

is **even**. (Here  $[x]$  denotes the largest integer smaller than or equal to  $x$ .)

**Solution.** Let  $f(n)$  denote the given equation. Then  $f(1) = 2$  which is even. Now suppose that  $f(n)$  is even for some  $n \geq 1$ . Then

$$\begin{aligned} f(n+1) &= \left[ \frac{n+1}{1} \right] + \left[ \frac{n+1}{2} \right] + \left[ \frac{n+1}{3} \right] + \cdots + \left[ \frac{n+1}{n+1} \right] + [\sqrt{n+1}] \\ &= \left[ \frac{n}{1} \right] + \left[ \frac{n}{2} \right] + \left[ \frac{n}{3} \right] + \cdots + \left[ \frac{n}{n} \right] + [\sqrt{n+1}] + \sigma(n+1), \end{aligned}$$

where  $\sigma(n+1)$  denotes the number of positive divisors of  $n+1$ . This follows from  $\left[ \frac{n+1}{k} \right] = \left[ \frac{n}{k} \right] + 1$  if  $k$  divides  $n+1$ , and  $\left[ \frac{n+1}{k} \right] = \left[ \frac{n}{k} \right]$  otherwise. Note that  $[\sqrt{n+1}] = [\sqrt{n}]$  unless  $n+1$  is a square, in which case  $[\sqrt{n+1}] = [\sqrt{n}] + 1$ . On the other hand  $\sigma(n+1)$  is odd if and only if  $n+1$  is a square. Therefore it follows that  $f(n+1) = f(n) + 2l$  for some integer  $l$ . This proves that  $f(n+1)$  is even.

Thus it follows by induction that  $f(n)$  is even for all natural number  $n$ .

3. Let  $a, b$  be natural numbers with  $ab > 2$ . Suppose that the sum of their greatest common divisor and least common multiple is divisible by  $a+b$ . Prove that the quotient is at most  $(a+b)/4$ . When is this quotient exactly equal to  $(a+b)/4$ ?

**Solution.** Let  $g$  and  $l$  denote the greatest common divisor and the least common multiple, respectively, of  $a$  and  $b$ . Then  $gl = ab$ . Therefore  $g+l \leq ab+1$ . Suppose that  $(g+l)/(a+b) > (a+b)/4$ . Then we have  $ab+1 > (a+b)^2/4$ , so we get  $(a-b)^2 < 4$ . Assuming,  $a \geq b$  we either have  $a = b$  or  $a = b+1$ . In the former case,  $g = l = a$  and the quotient is  $(g+l)/(a+b) = 1 \leq (a+b)/4$ . In the latter case,  $g = 1$  and  $l = b(b+1)$  so we get that  $2b+1$  divides  $b^2+b+1$ . Therefore  $2b+1$  divides  $4(b^2+b+1) - (2b+1)^2 = 3$  which implies that  $b = 1$  and  $a = 2$ , a contradiction to the given assumption that  $ab > 2$ . This shows that  $(g+l)/(a+b) \leq (a+b)/4$ . Note that for the equality to hold, we need that either  $a = b = 2$  or,  $(a-b)^2 = 4$  and  $g = 1, l = ab$ . The latter case happens if and only if  $a$  and  $b$  are two consecutive odd numbers. (If  $a = 2k+1$  and  $b = 2k-1$  then  $a+b = 4k$  divides  $ab+1 = 4k^2$  and the quotient is precisely  $(a+b)/4$ .)

4. Written on a blackboard is the polynomial  $x^2 + x + 2014$ . Calvin and Hobbes take turns alternatively (starting with Calvin) in the following game. During his turn, Calvin should either increase or decrease the coefficient of  $x$  by 1. And during his turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning strategy.

**Solution.** For  $i \geq 0$ , let  $f_i(x)$  denote the polynomial on the blackboard after Hobbes'  $i$ -th turn. We let Calvin decrease the coefficient of  $x$  by 1. Therefore  $f_{i+1}(2) = f_i(2) - 1$  or  $f_{i+1}(2) = f_i(2) - 3$  (depending on whether Hobbes increases or decreases the constant term). So for some  $i$ , we have  $0 \leq f_i(2) \leq 2$ . If  $f_i(2) = 0$  then Calvin has won the game. If  $f_i(2) = 2$  then Calvin wins the game by reducing the coefficient of  $x$  by 1. If  $f_i(2) = 1$  then  $f_{i+1}(2) = 0$  or  $f_{i+1}(2) = -2$ . In the former case, Calvin has won the game and in the latter case Calvin wins the game by increasing the coefficient of  $x$  by 1.

5. In an acute-angled triangle  $ABC$ , a point  $D$  lies on the segment  $BC$ . Let  $O_1, O_2$  denote the circumcentres of triangles  $ABD$  and  $ACD$ , respectively. Prove that the line joining the circumcentre of triangle  $ABC$  and the orthocentre of triangle  $O_1O_2D$  is parallel to  $BC$ .

**Solution.** Without loss of generality assume that  $\angle ADC \geq 90^\circ$ . Let  $O$  denote the circumcenter of triangle  $ABC$  and  $K$  the orthocentre of triangle  $O_1O_2D$ . We shall first show that the points  $O$  and  $K$  lie on the circumcircle of triangle  $AO_1O_2$ . Note that circumcircles of triangles  $ABD$  and  $ACD$  pass through the points  $A$  and  $D$ , so  $AD$  is perpendicular to  $O_1O_2$  and, triangle  $AO_1O_2$  is congruent to triangle  $DO_1O_2$ . In particular,  $\angle AO_1O_2 = \angle O_2O_1D = \angle B$  since  $O_2O_1$  is the perpendicular bisector of  $AD$ . On the other hand since  $OO_2$  is the perpendicular bisector of  $AC$  it follows that  $\angle AOO_2 = \angle B$ . This shows that  $O$  lies on the circumcircle of triangle  $AO_1O_2$ . Note also that, since  $AD$  is perpendicular to  $O_1O_2$ , we have  $\angle O_2KA = 90^\circ - \angle O_1O_2K = \angle O_2O_1D = \angle B$ . This proves that  $K$  also lies on the circumcircle of triangle  $AO_1O_2$ .

Therefore  $\angle AKO = 180^\circ - \angle AO_2O = \angle ADC$  and hence  $OK$  is parallel to  $BC$ .

**Remark.** The result is true even for an obtuse-angled triangle.

6. Let  $n$  be a natural number and  $X = \{1, 2, \dots, n\}$ . For subsets  $A$  and  $B$  of  $X$  we define  $A\Delta B$  to be the set of all those elements of  $X$  which belong to exactly one of  $A$  and  $B$ . Let  $\mathcal{F}$  be a collection of subsets of  $X$  such that for any two distinct elements  $A$  and  $B$  in  $\mathcal{F}$  the set  $A\Delta B$  has at least two elements. Show that  $\mathcal{F}$  has at most  $2^{n-1}$  elements. Find all such collections  $\mathcal{F}$  with  $2^{n-1}$  elements.

**Solution.** For each subset  $A$  of  $\{1, 2, \dots, n-1\}$ , we pair it with  $A \cup \{n\}$ . Note that for any such pair  $(A, B)$  not both  $A$  and  $B$  can be in  $\mathcal{F}$ . Since there are  $2^{n-1}$  such pairs it follows that  $\mathcal{F}$  can have at most  $2^{n-1}$  elements.

We shall show by induction on  $n$  that if  $|\mathcal{F}| = 2^{n-1}$  then  $\mathcal{F}$  contains either all the subsets with odd number of elements or all the subsets with even number of elements. The result is easy to see for  $n = 1$ . Suppose that the result is true for  $n = m - 1$ . We now consider the case  $n = m$ . Let  $\mathcal{F}_1$  be the set of those elements in  $\mathcal{F}$  which contain  $m$  and  $\mathcal{F}_2$  be the set of those elements which do not contain  $m$ . By induction,  $\mathcal{F}_2$  can have at most  $2^{m-2}$  elements. Further, for each element  $A$  of  $\mathcal{F}_1$  we consider  $A \setminus \{m\}$ . This new collection also satisfies the required property, so it follows that  $\mathcal{F}_1$  has at most  $2^{m-2}$  elements. Thus, if  $|\mathcal{F}| = 2^{m-1}$  then it follows that  $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{m-2}$ . Further, by induction hypothesis,  $\mathcal{F}_2$  contains all those subsets of  $\{1, 2, \dots, m-1\}$  with (say) even number of elements. It then follows that  $\mathcal{F}_1$  contains all those subsets of  $\{1, 2, \dots, m\}$  which contain the element  $m$  and which contains an even number of elements. This proves that  $\mathcal{F}$  contains either all the subsets with odd number of elements or all the subsets with even number of elements.

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