Regional Mathematical Olympiad-2016

Time: 3 hours **October 09, 2016**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let I be the incentre of ABC. Let AI extended intersect BC at F. Draw a line perpendicular to AI at I. Let it intersect AC at E. Prove that $IE = IF$.

2. Let a, b, c be positive real numbers such that
$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} = 1.$$
Prove that $abc \leq 1/8$.

3. For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$.

4. How many 6-digit natural numbers containing only the digits 1,2,3 are there in which 3 occurs exactly twice and the number is divisible by 9?

5. Let ABC be a right-angled triangle with $\angle B = 90°$. Let AD be the bisector of $\angle A$ with D on BC. Let the circumcircle of triangle ACD intersect AB again in E; and let the circumcircle of triangle ABD intersect AC again in F. Let K be the reflection of E in the line BC. Prove that $FK = BC$.

6. Show that the infinite arithmetic progression $\langle 1, 4, 7, 10, \ldots \rangle$ has infinitely many 3-term subsequences in harmonic progression such that for any two such triples $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ in harmonic progression, one has
$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}.$$

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