

# Regional Mathematical Olympiad-2016

Time: 3 hours

October 09, 2016

Instructions:

- Calculators (in any form) and protractors are not allowed.
  - Rulers and compasses are allowed.
  - Answer all the questions.
  - All questions carry equal marks. Maximum marks: 102.
  - Answer to each question should start on a new page. Clearly indicate the question number.
1. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $I$  be the incentre of  $ABC$ . Draw a line perpendicular to  $AI$  at  $I$ . Let it intersect the line  $CB$  at  $D$ . Prove that  $CI$  is perpendicular to  $AD$  and prove that  $ID = \sqrt{b(b-a)}$  where  $BC = a$  and  $CA = b$ .
  2. Let  $a, b, c$  be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that  $abc \leq 1/8$ .

3. For any natural number  $n$ , expressed in base 10, let  $S(n)$  denote the sum of all digits of  $n$ . Find all natural numbers  $n$  such that  $n = 2S(n)^2$ .
4. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.
5. Let  $ABC$  be a triangle with centroid  $G$ . Let the circumcircle of triangle  $AGB$  intersect the line  $BC$  in  $X$  different from  $B$ ; and the circumcircle of triangle  $AGC$  intersect the line  $BC$  in  $Y$  different from  $C$ . Prove that  $G$  is the centroid of triangle  $AXY$ .
6. Let  $\langle a_1, a_2, a_3, \dots \rangle$  be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.