

2014

BOOKLET NO.

TESTCODE : PHB

Afternoon

Questions : 6+8+8	Time : 2 hours
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On the answer-booklet write your Registration Number, Test Code, Number of this booklet, etc. in the appropriate places.

ATTENTION !

Please read the following very carefully before answering the test.

There are three parts. Part I is *compulsory* for all candidates and carries a credit of 30% of the total. Besides, each candidate has to choose *only one* of the Parts II & III and answer from that part as per instructions. The credit for each of these parts is 70% of the total.

**ALL ROUGH WORK MUST BE DONE ON THIS
BOOKLET AND/OR ON THE ANSWER-BOOKLET.
YOU ARE NOT ALLOWED TO USE CALCULATORS.**

STOP ! WAIT FOR THE SIGNAL TO START

Part I

Mathematical and Logical Reasoning

Answer all questions. All questions carry equal marks.

1. Let

$$f(x) = \frac{xe^{1/x} - x}{e^{1/x}}; \quad x \in \mathbb{R}.$$

Find $\lim_{x \rightarrow \infty} f(x)$.

2. Let

$$A = \frac{1}{2} \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

Find the eigenvalue of A^{2014} .

3. There are three balls labeled 1, 2, 3 and three boxes also labeled 1, 2, 3. Balls are placed at random into the boxes. Let X be the random variable that denotes the number of empty boxes. Find $E(X)$, the expectation of X .
4. A ball of unit mass is dropped from a height h . The frictional force of air is proportional to the velocity of the ball (with a constant of proportionality α). Show that the height of the ball in time t is

$$y(t) = h - \frac{g}{\alpha} \left[t - \frac{1}{\alpha} (1 - e^{-\alpha t}) \right],$$

where g is the acceleration due to gravity.

5. A satellite of mass m is in a stationary orbit above a point on the equator of the earth. The mass and radius of the earth are M and R respectively and its angular velocity about its own axis is ω . Find the height of the orbit above the ground.
6. A particle is constrained to move along the x -axis under the influence of the net force $F = -kx$ with amplitude A and frequency f , where k is a positive constant. When $x = \frac{A}{2}$, what is the speed of the particle?

Part II

Mathematics

Answer any five questions.

1. (a) A train with proper length L moves with speed $5c/13$ with respect to the ground, where c is the speed of light in vacuum. A ball is thrown from the back of the train towards the front. The speed of the ball with respect to the train is $c/3$. As viewed by someone from the ground, how much time does the ball spend in air and how much distance does it travel?
(b) A particle is moving under the influence of a central force such that $r \propto \frac{1}{\theta}$. Determine the potential energy as a function of r . The symbols have their usual meanings.
[7+7]
2. (a) Find the residue of the function $f(z) = \frac{\cot z \coth z}{z^3}$ at $z = 0$.
(b) Using a suitable contour evaluate $\int_0^{2\pi} \frac{d\theta}{3 - 2 \cos \theta + \sin \theta}$.
[7+7]
3. (a) Use the generating function for the Hermite polynomials to find $H_0(x)$, $H_1(x)$ and $H_2(x)$.
(b) Expand x^3 in a series of Hermite polynomials.
[7+7]
4. (a) Consider a concave mirror in the shape of a parabola with focus F whose equation is given by $y^2 = 4x$. Let P be a point source of light *inside* the parabola. Find Q on the parabola such that the ray PQ on reflection passes through the focus F .
(b) Let $\phi : (\mathbf{Q}, +) \rightarrow (\mathbf{Q}, +)$ be a homomorphism of the additive group of rationals into itself. Show that for some $\lambda \in \mathbf{Q}$

$$\phi(x) = \lambda x, \text{ for all } x \in \mathbf{Q}.$$

[7+7]

5. (a) Consider the following upper-triangular matrix A over Z_5 , the field of integers modulo 5:

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Show that A is invertible and find its inverse over Z_5 .

- (b) Consider the linear space $\mathcal{M}$ of all $n \times n$ matrices over $\mathbb{R}$. Show that $\mathcal{M}$ has a basis consisting of skew-symmetric and symmetric matrices.
- (c) Can there be a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that T maps $(1, 0, 1), (0, -1, 0), (1, 1, 1)$ to $(1, 0, 0), (0, 1, 0), (0, 1, 1)$ respectively? Justify.

[6+4+4]

6. (a) Find the Fourier coefficients corresponding to the function (period = 10)

$$\begin{aligned} f(x) &= 0, & -5 < x < 0 \\ &= 3, & 0 < x < 5 \end{aligned}$$

Also write the Fourier series corresponding to this function.

- (b) Let $p(x) = x^{2013} - x - 1$ be a real polynomial. Let $g(x)$ be a real-valued bounded continuous function. Show that there exists an $x_0 \in \mathbb{R}$ such that

$$p(x_0) = g(x_0).$$

[7+7]

7. (a) Let G be a group with no proper subgroup. Show that G is finite and hence cyclic.
- (b) Let G be the group of all 2×2 non-singular real matrices with matrix multiplication as the group operation. Give an example of a non-trivial normal subgroup of G .

[7+7]

8. (a) Determine the regions where the equation

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} + 3y^2 \frac{\partial u}{\partial x} = 0$$

is parabolic, hyperbolic and elliptic.

- (b) Find a recurrence formula and indicial equation for an infinite series solution around $x = 0$ for the differential equation

$$8x^2 y'' + 10xy' + (x - 1)y = 0.$$

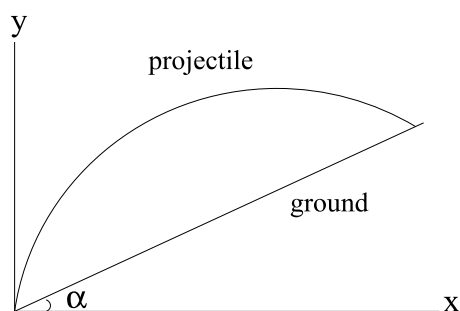
[7+7]

Part III

Physics

Answer any five questions.

1. (a) A projectile is fired uphill over the ground which slopes at an angle α to the horizontal (as shown in the figure). Find the direction in which it should be aimed to achieve the maximum range. (Hint: Use a relation between x and y coordinates where the projectile touches the ground.)



- (b) Consider the following transformation from a canonical set of phase space coordinates $\{q, p\}$ to another set of phase space coordinates $\{Q, P\}$,

$$Q = q^\alpha \cos(\beta p), \quad P = q^\alpha \sin(\beta p)$$

where α and β are constants. Show that the transformation is canonical for $\alpha = \frac{1}{2}$ and $\beta = 2$.

[7+7]

2. (a) Consider the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 - w^2 x^2)e^{\gamma t}$$

for the motion of a particle of mass m in one dimension. The constants m , γ and w are real and positive.

- Derive the equation of motion.
- Find the canonical momentum, and from this construct the Hamiltonian function.

- (b) A person standing at the rear of a railroad car fires a bullet towards the front of the car. The speed of the bullet, as measured in the frame of the car, is $0.5c$ (where c is the speed of light in vacuum) and the proper length of the car is 400m. The train is moving at $0.6c$ as measured by observers in the ground. For the ground observers, find
- the length of the railroad car,
 - the speed of the bullet,
 - the time required for the bullet to reach the front of the car.

$$[(3+4)+(2+2+3)]$$

3. (a) Consider a simple harmonic oscillator in one dimension with the Hamiltonian

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

where a and $a^\dagger$ are the annihilation and creation operators respectively and the other symbols have their usual meanings. The ket vector of the harmonic oscillator at $t = 0$ is given by

$$|\psi(0)\rangle = N (|0\rangle + 2|1\rangle + 3|2\rangle)$$

where N is the normalization constant and $|n\rangle$ is the eigenket of corresponding energy eigenvalue $E_n = \hbar\omega (n + \frac{1}{2})$.

- Find the normalization constant N .
 - Calculate the probability of finding the energy to be $\frac{3}{2}\hbar\omega$ on energy measurement.
 - Find the ket vector $|\psi(t)\rangle$ at time t and calculate expectation value of the energy for this ket vector.
- (b) Consider the following ket vector of the harmonic oscillator

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\mu|^2} \frac{\mu^n}{\sqrt{n!}} |n\rangle$$

where $|n\rangle$ is an eigenket of the Hamiltonian with eigenvalue $(n + \frac{1}{2})\hbar\omega$ and μ is a complex number. Show that

$$\hat{a}|\phi\rangle = \mu|\phi\rangle$$

where $\hat{a}$ is the harmonic oscillator annihilation operator.

$$[(1+2+3)+8]$$

4. (a) Consider N fixed non-interacting magnetic moments each of magnitude μ_0 . The system is in thermal equilibrium at temperature T and is in a uniform external magnetic field B . Each magnetic moment can be oriented only parallel or antiparallel to B . Calculate
- the partition function,
 - the specific heat.
- (b) For a degenerate, spin $\frac{1}{2}$, non-interacting Fermi gas at zero temperature, find an expression for the energy of a system of N such particles confined to a volume V . Assume the particles are non-relativistic.

[(5+5)+4]

5. (a) A sphere of radius R_1 has charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located at a distance a from the centre.
- Find the field $\mathbf{E}$ at the centre of the hollow sphere.
 - Find the potential ϕ at the same point.
- (b) An electric charge Q is uniformly distributed over the surface of a sphere of radius r . Show that the force on a small charge element dq is radial and outward and is given by

$$d\mathbf{F} = \frac{1}{2}\mathbf{E}dq$$

where $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$ is the electric field at the surface of the sphere. ϵ_0 is the permittivity of the free space.

[(4+4)+6]

6. (a) A particle moves in a time-independent electric field $\mathbf{E} = -\nabla\phi$ and any magnetic field $\mathbf{B}$. Using Lorentz force law, can you show that the energy of the particle is constant?
- (b) A particle moves along the x -axis in the electric field $\mathbf{E} = A \exp^{-t/\tau} \hat{\mathbf{i}}$ (where A and τ are constants) and the magnetic field is zero along the x -axis. Find $x(t)$ with the initial conditions $x(0) = \dot{x}(0) = 0$.
- (c) A one-dimensional quantum harmonic oscillator (with ground state energy $\hbar\omega/2$) is in thermal equilibrium with a heat

bath at temperature T . Determine the mean value of the oscillator's energy, $\langle E \rangle$, as a function of T .

[4+4+6]

7. (a) A particle moves in a central potential $V(r) = -\frac{g^2}{r^{3/2}}$, where g is a constant. Using the normalized ground state wave function of the Hydrogenic atom $\psi(r) = \left(\frac{k^3}{8\pi}\right)^{1/2} e^{-kr/2}$, find out the upper bound of the lowest s -state energy.
- (b) Consider a relativistic field theory involving two scalar fields ϕ and ψ with ϕ having mass parameter m and ψ being massless. The interaction term is $g\phi^2\psi^2$, where g is the coupling constant.
- Write down the action of the system in two space and one time dimensions.
 - Derive the equations of motion. Write down the effective masses for ψ and ϕ .
 - Consider the process where one ϕ particle and one ψ particle of momenta p_1^μ and p_2^μ respectively scatters to one ϕ particle and one ψ particle of momenta q_1^μ and q_2^μ respectively. Draw Feynman diagrams for the lowest order and next to lowest order processes with proper labeling of the momenta.

[7+(1+3+3)]

8. (a) Explain why the following processes are not observed in nature. Discuss any four of the seven options. (The symbols carry their usual meaning.)

$$\begin{aligned}
 p &\rightarrow e^+ + \pi^0 \\
 \Lambda^0 &\rightarrow K^0 + \pi^0 \\
 p + \bar{p} &\rightarrow \Lambda^0 + \Lambda^0 \\
 \Lambda^0 &\rightarrow K^+ + K^- \\
 n &\rightarrow p + e^- \\
 p &\rightarrow e^+ + \nu_e \\
 \mu^+ &\rightarrow e^+ + \gamma
 \end{aligned}$$

- (b) Very high energy protons in cosmic rays can lose energy through a collision process

$$p + \gamma \rightarrow p + \pi.$$

The typical energy radiated in this process is 2.73 K. How energetic need a cosmic ray proton be to be above the threshold for this reaction? Given that the Boltzmann constant $k = 8.6 \times 10^{-5}$ eV/K, and the masses of proton and pion are 0.938 GeV and 0.140 GeV respectively.

$$[(4 \times 2)+6]$$