2016

BOOKLET NO. TEST CODE: QRB

Afternoon **Time:2 Hours**

Read the following carefully before answering the test.

Write your Registration Number, Test Code, Booklet No. etc., in the appropriate places on the answer booklet.

The question paper is divided into **two** groups: **Group A** and **Group B**.

- Group A has **one** question on **Mathematics**, **two** on **Probability** and **one** on **Statistics**.
- Group B has totally **eight** questions - **two on each of Operations Research, Reliability, Statistical Quality Control and Quality Management**.

Each question carries 24 marks. You have to answer two questions from Group A and three questions from Group B. Marks are specified against the questions.

- Partial credit may be given to partial answers.
- Full credit may be given to complete and rigorous arguments.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER-BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS IN ANY FORM.

STOP! WAIT FOR THE SIGNAL TO START.

Mathematics

1. (a) Let $A = CC^t$ and $B = DD^t$, where C and D are real matrices so that AB is well defined. Prove that if trace of AB is equal to 0, then $C^tD = 0$. Here X^t stands for the transpose of the matrix X. (8)

   (b) Define $f(x, y) = 2x^2 + 3y + 4$ for any point $(x,\ y)$ on the line segment joining the points $(0,\ 0)$ and $(-4,\ 3)$. Transform this function into a univariate function of nonnegative variable and sketch the graph of the transformed function. (8)

   (c) Let A be a fixed real matrix with 3 rows. For any 3×1 real vector b, define $S(b) = \{x : Ax = b\}$. Assume that $S(b_i) \neq \emptyset$ for $i = 1, 2, 3$, where $b_1 = (1, 2, 3)^t$, and $b_2 = (3, 2, 1)^t$ and $b_3 = (4, 4, 4)^t$. Further assume that for any b, $S(b) \neq \emptyset$ implies that b is a linear combination of b_1, b_2 and b_3. Find the rank of the matrix A and justify your answer. (8)

Probability

2. Each human has two chromosomes. While all males have one X chromosome inherited from mother and the Y chromosome inherited from father, all females have two X chromosomes inherited one from mother and one from father. Call an X chromosome bad if it leads to a disease called haemophilia, else call it good. It is known that ladies having children have at most one bad chromosome. Consider a lady whose brother has a bad chromosome but her father has a good chromosome. Find the probability that the lady is a carrier, that is, the lady has a bad chromosome if

   (a) the lady has at most one bad chromosome

   (b) the lady has only one child, a son with a bad X chromosome

   (c) the lady has only one child, a son with a good X chromosome

(d) the lady has only two children, sons, both with good X chromosomes

(e) the lady has only two children, sons, both with bad X chromosomes

(f) the lady has only one child, a daughter. $(4 \times 6 = 24)$

3. (a) A basket contains 50 items of which 20 are defective. Two items are drawn, one after the other, without replacement. Define $Y_i = 1$ if i^{th} item drawn is defective, $Y_i = 0$ if it is good, $i = 1, 2$.

   i. Show that Y_1 and Y_2 are identically distributed. (4)

   ii. Are Y_1 and Y_2 independent? (2)

   iii. What is the expected value of $Y_1 + Y_2$? (2)

   (b) An urn contains n strings. Two ends are picked up randomly, the ends are tied and the resulting string/loop is dropped back into the urn. This process is repeated n times (because there are a total of $2n$ ends to begin with). The number of loops formed at the end of such a process, X, is a random variable. Find the probability that the k^{th} knot results into a loop. Hence or otherwise, find out the expectation of X. (8)

   (c) A software code has N lines and each line is either defective or good. The problem is to estimate the number of defective lines. The code is independently tested by two engineers - one of them experienced and the other is fresher. Each engineer marks each line as good or defective, separately. Assume that none of them marks a good line as defective. Let x be the number of lines marked as defective by the first engineer, y be the number of lines marked as defective by the second engineer, and z be the number of lines marked as defective by both engineers. Estimate the number of defective lines and explain your method. (8)

Statistics

4. (a) Let $X_1 \sim N(0,1)$ and define

$$X_2 = \begin{cases} -X_1 & \text{if } -1 \leq X_1 \leq 1 \\ X_1 & \text{otherwise} \end{cases}$$

Prove that X_2 also has the same distribution as that of X_1. Are X_1 and X_2 independent? $(6+2=8)$

(b) Let $x_1,\ x_2,\ x_3$ be a sample from Bernoulli random variable with parameter p. Is $x_1+2x_2+x_3$ a sufficient statistic for p? Is $h(x_1,x_2,x_3) = (x_2,\ x_1+2x_2+x_3)$ a sufficient statistic? $(6+4=10)$

(c) Let $x_1,\ x_2,\ x_3$ be a sample from Bernoulli random variable with parameter p, $p \in \{\frac{1}{3},\ \frac{2}{3}\}$. Is $x_1+x_2+x_3$ complete? (6)

Operations Research

5. Let

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 & -3 \\ -1 & -2 & 2 & 1 & 3 \end{bmatrix}$$

and let $S(b) = \{x : Ax = b,\ x \geq 0\}$, where $b = (b_1,\ b_2)^t$ is any 2×1 real vector.

(a) Show the columns of A in $\mathbf{R}^2$ and draw the picture of the cone generated by third, fourth and fifth columns of A. (6)

(b) Is there a b for which $S(b) = \emptyset$? Justify your answer. (6)

(c) Find the extreme points and all linearly independent directions of $S(b)$ for $b = (-7,\ 5)^t$. For this b, does the problem

Minimize $2x_1 - 3x_2 + x_3 + 4x_4$, subject to $x = (x_1, x_2, \ldots, x_5)^t \in S(b)$

have an optimal solution? If it has, find it, else justify. (12)

6. (a) A business executive must make the four round-trips between Dallas and Atlanta. The trips are listed in the table below. A round-trip fare from Dallas is \$400. A 25% discount is granted if the dates of arrival and departure of a ticket span a weekend (Saturday and Sunday). If the gap between the dates of a round-trip is more than 21 days, the discount is increased to 30%. A one-way ticket between Dallas and Atlanta (either direction) costs \$250. Formulate the problem to determine how the executive should purchase the tickets. Find the optimal solution.

Executive Visit Schedule

Departures from Dallas	Return dates from Atlanta
Monday Jun 3rd	Friday Jun 7
Monday Jun 10	Wednesday Jun 12
Monday Jun 17	Friday Jun 21
Tuesday Jun 25	Friday Jun 28

$(9 + 3 = 12)$

(b) Consider the following linear fractional programming problem:

Minimize $f(x) = \dfrac{c^t x + \alpha}{d^t x + \beta}$ subject to $Ax \leq b$, $x \geq 0$, where $d^t x + \beta \neq 0$, $A \in \mathbf{R}^{n \times n}$, $b \in \mathbf{R}^m$, $c, d \in \mathbf{R}^n$, $\alpha, \beta \in \mathbf{R}$.

Write the KKT condition of the given problem. State under what condition, a KKT point is an optimal point. Formulate a linear programming problem based on the above condition. (12)

Reliability

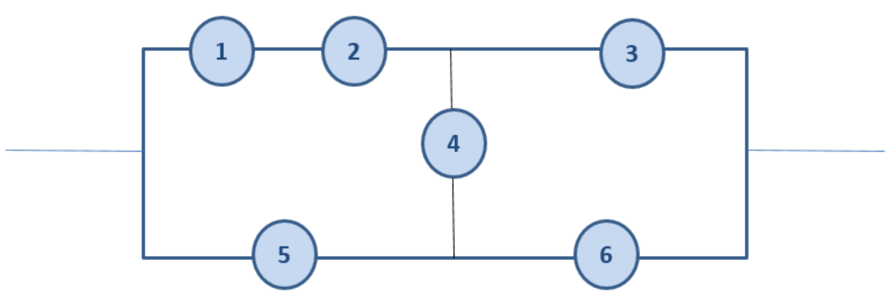
7. (a) A diode may fail due to either open or short failure modes. Let the failure density due to open mode be f_o and that due to short mode be f_s. Assume that the failure density f of the diode is given by $f(t) = pf_o(t) + (1-p)f_s(t),\ t \geq 0$, where p is a number greater than 0 and less than 1.

   i. Explain the meaning of p. (4)

   ii. Derive the reliability and hazard functions of the diode under the assumption that f_o is exponential with failure rate λ and that f_s is exponential with failure rate μ. $(3+3=6)$

   (b) In a static system, let X be the random variable representing the stress on the system and let Y be the random variable representing the strength of the system. When both X and Y have exponential distribution, find the reliability of the system. What should be the mean strength compared to the mean stress in order to achieve a reliability of 0.95? (6)

   (c) Consider the following structure of a system.



   i. Find the minimal cut sets and minimal path sets. (4)

   ii. Determine the structure function using either the minimal path sets or the minimal cut sets. (4)

8. (a) Explain what is censoring in life testing. (6)

   (b) A company manufacturing CFL bulbs, takes five bulbs randomly from its previous day's production and puts them on test slots every day morning and continuously burn them up to 30 days. Due to lack of modern facilities, the bulbs are only observed every day morning at 7 AM to see if they are still burning. This is done just before placing the new bulbs on test on the free slots available. In this context, explain what you understand by left censoring, right censoring and arbitrary censoring. Do you get to observe all these three types censored data in this testing of bulbs? (6)

   (c) Assume that the total number of test slots available for life testing is N so that there is a possibility of not being able to place all five bulbs on test each day morning. Consider the number of test slots available each day at morning at 7 AM for placing the new bulbs on test. Model the system as a Markov Chain. Assuming that the bulb life follows exponential distribution with rate of failure as λ per hour, obtain the one-step transition matrix. Explain the steady state probabilities and express the probability of having test slots for the five new bulbs every day morning in terms of the steady state probabilities. (12)

Statistical Quality Control

9. (a) In a plant producing pumps, one of the critical components is piston, the production process of which is found to be in statistical control. A key characteristic of the piston is its outer diameter (OD), the standard deviation of which is found to be 0.2 mm based on an analysis of the past data. The upper and lower specification limits of the piston OD are 6.5 mm and 5.5 mm respectively. The costs of rejection due to under-specification and over-specification have been worked out to be Rs. 1000 per piston and Rs. 10 per piston respectively. The profit per piston is Rs. 100 provided the piston conforms to the specifications. Considering the underlying distribution of OD to be normal, arrive at the economic process centre with appropriate formulation of the problem. ($\log_e 10 = 2.302585$) (12)

   (b) A process of interest is in a state of statistical control and generates continuous random measurements for a measured characteristic X from a normal distribution with a known expected value μ and a known standard deviation σ. Consider that the measured characteristic has lower and upper specification limits (LSL and USL respectively). For such a process prove that

$$C_{pk} = (1 - k)C_p$$

   where C_p is process potential index, C_{pk} is process performance index and k is the process centering index. (8)

   (c) One wishes to design a control chart for non-conformities per unit with L-sigma limits. Find the expression for the minimum number of inspection units that will yield a positive lower control limit for this chart. (4)

10. (a) Consider a normally distributed process with $\mu = 20$ and $\sigma = 2$. The lower and upper specification limits for the associated quality characteristic are LSL $= 8$ and USL $= 32$ respectively. Set-up the parameters of an appropriate control chart for monitoring the mean with sample size $n = 4$, if the mean is allowed to drift to the tune of 1.5σ. (8)

(b) Assume that the quality characteristic of interest $X \sim N(\mu, \sigma^2)$. Let the lower and upper specifications for X be $[L, U]$ and the potential process capability index of the process is C_p. If the process is operating at the mid-point of the specification limits then prove that the expected proportion of conforming items is given by $2\Phi(3C_p) - 1$. (8)

(c) Design different possible double sampling plans without curtailment with $p_1 = 0.03$, $\alpha = 0.05$, $p_2 = 0.12$, $\beta = 0.10$ and $n_2 = 2n_1$ by consulting the following Grubbs table. The symbols have their usual meanings. (8)

Grubbs Table for Double Sampling Plan

Plan	Ratio of proportion defective for low P_a to that for high P_a	c_1	c_2	pn_1	
				$1 - \alpha = 0.95$	$\beta = 0.10$
1	14.50	0	1	0.16	2.32
2	8.07	0	2	0.30	2.42
3	6.48	1	3	0.60	3.89
4	5.39	0	3	0.49	2.64
5	5.09	1	4	0.77	3.92
6	4.31	0	4	0.68	2.93
7	4.19	1	5	0.96	4.02
8	3.60	1	6	1.16	4.17
9	3.26	2	8	1.68	5.47
10	2.96	3	10	2.27	6.72

Quality Management

11. (a) In a paint manufacturing plant supervisors used to avoid reporting any batch of paint that did not meet specifications. In fact, the matter went to such an extreme level that in order to hide some non-conforming paint, one supervisor directed his workers to dig a hole in the yard and bury any non-conforming paint.

    i. Explain the phenomenon from quality culture point of view considering various hierarchies of human needs and associated forms of quality motivation. (14)

    ii. Give one striking example for positive quality practice originated from prevailing positive quality culture in an organization. (4)

    iii. How would you define Theory X and Theory Y of management that can be triggered from the cited situation? (6)

12. (a) In which phases of DMAIC in Six Sigma FMEA can be used? What is the purpose of FMEA in each of these phases? In addition to FMEA, describe other techniques that one can use in different phases of Six Sigma. $(2 + 3 + 3 = 8)$

    (b) Explain the critical factors for successful implementation of Six Sigma. (6)

    (c) DPMO is a commonly used metric for evaluating process performance in Six Sigma. However, there are certain drawbacks to use DPMO to compare different dissimilar processes – discuss. (10)

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