2014

BOOKLET NO. TEST CODE: **QRB**

Afternoon **Time: 2 hours**

Read the following carefully before answering the test.

Write your Name, Registration Number, Test Code, Booklet No. etc., in the appropriate places on the answer-booklet.

The question paper is divided into **two** groups: **Group A** & **Group B**.

- Group A has questions on **Mathematics** and **Statistics**.
- Group B has questions on **Operations Research, Quality Management, Reliability** and **Statistical Quality Control**.

Each question in each group carries a total of 20 marks. You have to answer any **six questions**, taking **at least two questions from each group**. Marks allotted to each question is given within [ ].

- **Partial credit may be given for partial answer.**
- **Full credit will be given for complete and rigorous arguments.**

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER-BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS IN ANY FORM.

STOP! WAIT FOR THE SIGNAL TO START.

Group A

Mathematics

1. (a) Let p be a prime number greater than 3. Show that $p^2 - 1$ is divisible by 24. [5]

   (b) Let f be a real valued thrice differentiable function defined by $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$, where x is real. Find the value of $f(2)$. [6]

   (c) Consider functions from $A = \{0, 1, 2\}$ to $B = \{0, 1, 2, \ldots, 7\}$. Find the number of functions f satisfying $f(p) < f(q)$ for $p < q$, and hence the number of functions satisfying $f(p) \leq f(q)$ for $p < q$. [9]

2. (a) A point is randomly chosen on each side of a unit square. Let a, b, c and d be the sides (lengths) of the quadrilateral formed by these four points. Show that

   (i) $2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$ [5]

   (ii) $2\sqrt{2} \leq a + b + c + d \leq 4$. [5]

   (b) Let $x_1,\ x_2,\ \ldots, x_n$ be the marks of n students with $\bar{x} = \left(\sum_{i=1}^{n} x_i\right)/n$ and $V = \left(\sum_{i=1}^{n} (x_i - \bar{x})^2\right)/n$. Show that for any positive number k, the number of x_i's satisfying $|x_i - \bar{x}| \geq k$ (or equivalently $(x_i - \bar{x})^2 \geq k^2$) is at most nV/k^2. [10]

3. Let k be a fixed positive integer greater than 5. Let $A = ((a_{ij}))$ be the $k \times k$ matrix defined by $a_{ij} = i$ if $i = j$ and $a_{ij} = 1$ otherwise. Let A_n denote the n^{th} root of A; that is, n^{th} power of A_n is A.

(a) Prove that A is a positive definite matrix and that its eigen values are positive. [7]

(b) Assuming that A_n exists for every n, what is $\lim_{n\to\infty} A_n$? [3]

(c) Prove or disprove: A_n exists for every positive integer n. [7]

(d) If A_n exists for any n, explain the procedure for computing it. [3]

Statistics

4. (a) Suppose that two positive numbers are chosen randomly from 1 to 50. What is the probability that their difference is divisible by 3? [7]

(b) The probability p_k that a family has k children is given by

$$p_k = \begin{cases} a & \text{for } k = 0, 1; \\ (1-2a)^{-(k-1)} & \text{for } k \geqslant 2, \end{cases}$$

where $a < 1/2$. Assuming that the probability of a boy child is the same as that of a girl child in a family, compute the conditional probability that the family has only two children given that the family has two boys. [6]

(c) Consider a Markov Chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $S = \{0, 1, 2, \ldots\}$ such that

$$P[X_n = j | X_{n-1} = i] = \begin{cases} p & \text{if } j = i+1; \\ 1-p & \text{if } j = i. \end{cases}$$

i) Derive $P[X_n = 3 | X_0 = 0]$, for $n = 0, 1, 2, \ldots$. [4]

ii) Comment on the irreducibility of the Markov chain with proper justification. [1]

iii) Comment on the stationary distribution of the Markov Chain with proper justification. [2]

5. (a) Suppose $(X_1, X_2, \ldots, X_n) \sim$ iid $N(\mu, 1)$.
Define

$$Y_i = \begin{cases} 1 & \text{if } X_i > a; \\ 0 & \text{otherwise.} \end{cases}$$

If X_i's are unobservable but Y_i's are observed, find the maximum likelihood estimator (MLE) of μ, in terms of the cdf of $N(0,1)$ when a is known. Will the MLE increase if $Var(X_i)$ is increased? Justify. [4+2]

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the pdf

$$f(x|\theta) = \frac{\theta}{x^2}, \quad 0 < \theta \leqslant x < \infty.$$

i) Find the MLE of θ. Show that the MLE is consistent. [3+4]

ii) Find the distribution of Z where $Z = 2\sum_{i=1}^{n}(\ln X_i - \ln\theta)$. [4]

iii) Hence find the $100(1-\alpha)\%$ confidence interval of θ. [3]

6. (a) Suppose $(Y, Z)' \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and the conditional distribution of Ygiven $Z = z$ is $N(-z, 1)$. Also it is given that $E(Z|Y = y) = -\frac{1}{3} - \frac{1}{3}y$. Then determine $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. [8]

(b) Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_5$ be a random sample from $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\mathbf{\Sigma} = \begin{pmatrix} 3 & 0.1 \\ 0.1 & 1 \end{pmatrix}.$$

Suppose that the sample mean is $\bar{X} = (1, 0)'$. Test the hypothesis

$$H_0 : \boldsymbol{\mu} = (0, 0)' \quad \text{Vs} \quad H_1 : \boldsymbol{\mu} \neq (0, 0)'$$

at 5% level of significance. [Given that, $\chi^2_{0.05,2} = 5.99$]. [4]

(c) Consider the following fixed effect linear model:

$$\begin{aligned} y_1 &= \alpha + \epsilon_1 \\ y_2 &= y_1 + \alpha - \beta + \epsilon_2 \\ y_3 &= y_2 + \alpha + \epsilon_3 \\ y_4 &= y_3 + \alpha + \beta + \epsilon_4 \\ y_5 &= y_4 + \alpha + 2\beta + \epsilon_5, \end{aligned}$$

where $y_1, y_2, \ldots, y_5$ are the observations; α, β are the parameters and ϵ_i's are uncorrelated random variables with $E(\epsilon_i) = 0$ and $Var(\epsilon_i) = \sigma^2, i = 1\,(1)\,5$. Find the least square estimates of α and β based on the observations $y_1 = 1$, $y_2 = 2, y_3 = 3, y_4 = 4, y_5 = 6$. [8]

Group B

Operations Research

7. (a) Write the dual of the following linear programming problem:

$$\begin{aligned} &\text{Minimize } 3x_1 - 4x_2 + 5x_3 \\ &\text{subject to} \\ &\begin{array}{rcl} x_1 - \ x_2 + \ x_3 & \geq & 2 \\ 2x_1 + \ x_2 - \ x_3 & = & 7 \\ x_1 - 2x_2 + 3x_3 & \leq & 2 \end{array} \\ &x_1 \geq 0, x_2 \leq 0, x_3 \in \mathbf{R}. \end{aligned}$$

[7]

(b) State the fundamental theorem of duality. [3]

(c) Using the above theorem, show, without actually solving it, that the following problem has an optimal solution:

$$\begin{aligned} &\text{Maximize } 5x_1 - 3x_2 + 9x_3 \\ &\text{subject to} \\ &\begin{array}{rcl} 2x_1 - 3x_2 + 5x_3 & \leq & 10 \\ -\ x_1 + 2x_2 + \ x_3 & \leq & 11 \\ 4x_1 - 2x_2 + 3x_3 & \leq & 12 \end{array} \\ &x_1, x_2, x_3 \geq 0. \end{aligned}$$

[10]

8. Consider the following problem:

$$\begin{aligned} &\text{Minimize } x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1 - x_4 \\ &\text{subject to} \\ &2x_1 + x_2 + x_3 + x_4 \leq 7 \\ &x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

(a) Let $\bar{x} = [2, 2, 1, 0]^t$. Is $\bar{x}$ a local optimum solution? If not, find a feasible direction and the next point along the feasible direction. [12]

(b) Find the point $\hat{x}$ at which $\nabla f(\hat{x}) = 0$. Hence, find optimal solution of the problem. Is it the only optimal solution? Justify your answer. [8]

9. Consider the optimization problem:

$$\text{Minimize } c^t x \text{ subject to } e^t x = 0,$$

where $c,\ x,\ e$ are column vectors in $\mathbf{R^n}$, e is the vector of 1s and c is a fixed vector.

(a) Formulate the problem as a linear programming problem with non-negative variables. [3]

(b) State the condition under which the problem has an optimal solution. [4]

(c) A *perfect set* is any set S of feasible solutions x to the minimization problem above satisfying (i) $x^t x = 1$ and (ii) $x^t y = 0$ for all $x,\ y \in S$ and $x \neq y$. Call a perfect set a maximal perfect set if it has largest number of elements among all perfect sets. What is the number of elements in a maximal perfect set? [6]

(d) Describe a method of finding elements of a maximal perfect set solving a set of linear programming problems iteratively. [7]

10. Prepare the matrix by putting a cross mark ($\times$) for the problem-solving tools most likely to be used at each of the DMAIC stages as shown in the following table. The solution for serial number 1 has been shown to you as an example.

Table 1: Likely Tools in the DMAIC Model

Sl.No	**Tools**	**Stages**				
		D	**M**	**A**	**I**	**C**
1	Run chart		$\times$		$\times$	$\times$
2	Brainstorming					
3	Poka-yoke					
4	Cause and effect diagram					
5	Design of experiments					
6	Pareto analysis					
7	Gage R & R study					
8	Project charter					
9	Anticipated bottom-line benefit					
10	SIPOC diagram					
11	Detailed process flow-chart					
12	Force field analysis					
13	Project prioritization matrix					
14	Solution prioritization matrix					
15	Scatter diagram					
16	Control plan					
17	Hypothesis testing					
18	ANOVA					
19	Box plot					
20	Control chart					
21	Sigma level computation					

[20]

11. (a) A process produces a total of 400 units and has a total of 75 defects. A unit may contain multiple defects. Determine the expected number of units it would take to produce 100 conforming units. It is given that $e^{-0.1875} = 0.829029$. [5]

(b) A process had a rolled throughput yield (Y_{RT}) of 0.47774 for 10 operations. Mention the procedure and related expressions to determine normalized yield (Y_{NORM}), defects per normalized unit (DPU_{NORM}) and $Z_{BENCHMARK}$. [4]

(c) While solving a problem using Six Sigma approach, in one of such phases, it was observed that for different product types different amount of scrap was being produced as given below:

Table 2: Scrap for Different Product Type

Product Type	**Scrap**
D_{24}	28
D_{30}	12
D_{16}	10
D_{20}	9
D_{14}	8
D_{28}	8
D_{18}	6
D_{8}	6
D_{10}	5
D_{12}	4
D_{6}	4

Use an appropriate tool to find out the product type(s) that one should target in order to eliminate at least 40% of the scarp that are currently being generated. [4]

(d) A gage repeatability and reproducibility (R & R) study was carried out with two operators measuring 10 samples taken from a wide range of the product that is made on a regular basis. Each sample is measured thrice by each operator. The identity of the samples remains undisclosed to the operators.

i) Write down the model for carrying out the Analysis of Variance (ANOVA) for the above experiment. [2]

ii) Write down the ANOVA table. [3]

iii) How would you test the homogeneity of the samples? [2]

Reliability

12. (a) Consider a 2-out-of-3 system of independent components each having $\exp(\lambda)$ life distribution.

(i) Derive reliability function of the system life and prove that the system life distribution is Increasing Failure Rate (IFR). [3+4]

(ii) Consider another 2-out-of-3 system with independent components each having $\exp(\lambda)$ life distribution, which is independent of the first 2-out-of-3 system, is arranged in parallel with the first system. Give an expression for reliability of the resulting system. [3]

(b) Suppose there are two independent components C_1 and C_2 each having $\exp(\lambda)$ life distribution.

(i) Consider a parallel system with the above components. Upon failure of one component, hazard rate of the other changes to 2λ. Find the reliability function of the system. [6]

(ii) Consider a single component system, where component C_1 is functioning and component C_2 is in standby. Component C_2 is put into operation after failure of component C_1 . Find the hazard rate of the system. [4]

13. (a) Consider a lifetime T with the hazard function

$$\lambda(t) = \begin{cases} \lambda_1 & \text{for } 0 \le t < t_1 \\ \lambda_2 & \text{for } t_1 \le t < \infty. \end{cases}$$

Find the mean residual life for $t \ge t_1$ and $t < t_1$. Hence find the mean lifetime. [6+2]

(b) Suppose a device has lifetime distribution T with the p.d.f.

$$f(t) = 2\lambda t e^{-\lambda t^2}, \quad t > 0, \lambda > 0.$$

(i) Find the distribution of T^2. [3]

(ii) Suppose n identical devices are put on a life test at time $t = 0$. The life test is continued until a pre-fixed number r ($\le n$) of the devices have failed (the remaining are censored at the r-th failure time). Derive the maximum likelihood estimator (MLE) of λ. Check whether the MLE of λ is unbiased. [4+5]

14. Consider a single unit system maintained with independent components having $\exp(\lambda)$ life distribution.

(a) Prove that the probability of at least n replacements by the first 500 hours of operation is

$$\int_0^{500} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} dt.$$

[4]

(b) Find the probability that there are $d_1 (< d)$ replacements in the first 200 hours given that d replacements are observed by the first 500 hours. [4]

(c) Find the probability that there will be d_2 replacements in the next 200 hours (after observing for 500 hours) given that there are d replacements in the first 500 hours. [4]

(d) Find the MLE of λ based on the information that there are d replacements in 500 hours and the distribution of this MLE. [3+5]

Statistical Quality Control

15. (a) The thickness (X) of a printed circuit board is an important quality characteristic and $\bar{X}$ chart is being used for monitoring of its mean. A newly recruited quality manager argued that the currently used parameters of the $\bar{X}$ chart, i.e., sample size, sampling interval and control limits may not be appropriate. These should be selected in such a way that the expected net income per production cycle is maximized.

It is observed that the process standard deviation σ remains unchanged. The time required for collection of samples and interpretation of results is proportional to the sample size. Only a single assignable cause of magnitude δ can occur at random according to a Poisson process with an intensity of λ occurrences per hour. Since only a single assignable cause occurs, a fixed time is required for searching and its elimination. Derive the length of a production cycle under the assumption that the process operation is stopped during the search for the assignable cause. [15]

(b) An engineer is constructing an $\bar{X} - R$ control chart for outside diameter (OD) of bolts based on samples of size 5. The sample statistics thus obtained are $\bar{R} = 0.023$ and $\bar{\bar{X}} = 5.001$. Since the process exhibits good control he wants to reduce the sample size to 3. The pertinent constants are given below:

Table 3: Control Chart Constants

n	d_2	A_2	D_3	D_4
3	1.693	1.023	0	2.575
5	2.326	0.577	0	2.115

Find the new control limits for $\bar{X}$ as well as R charts. [5]

16. (a) Consider a double sampling plan with lot size $N = 2000$ and parameters $n_1 = 30, c_1 = 1, n_2 = 60$ and $c_2 = 3$. The lot fraction nonconforming value is found to be 0.05 for a series of lots. Evaluate the probability of acceptance for the combined samples. (Instead of exact calculations appropriate expressions will be appreciated.) [12]

    (b) Explain the effect of measurement error in estimating process capability. How do you estimate C_p when measurement error cannot be ignored? [2+6]

17. (a) A manufacturer of an electronics gadget buys memory devices in lots of size N from a vendor. The vendor has a long record of good quality performance with an average fraction defective p. The quality control department decided to use a single sampling plan with sample size n and acceptance number 0. The vendor complaints to the manufacturer that very often lots of acceptable quality get rejected since the sampling plan is quite stringent.

    Consequently the manufacturer modified the sampling inspection plan as follows:

From each lot, select a sample of size n and observe the number of defectives d. If $d = 0$, accept the lot; if $d = 1$, accept the lot provided there have been no defective in the previous consecutive "i" lots. If $d \geqslant 2$, reject the lot.

Determine the probability of acceptance of a lot under the modified sampling inspection plan. [5]

(b) An $\bar{X}$ chart is used to control the mean of a normally distributed quality characteristic. It has a centre line of 100 with 3σ control limits and is based on sample size of 9. The process standard deviation is known to be 6. If the process mean shifts from 100 to 108 what is the probability that the shift will be detected on the second subsequent sample following the shift? [Given that $\Phi(1) = 0.84134$]. [5]

(c) A company manufactures personal computers by assembling various components. As the process does not result in the natural formation of lot, it is decided to implement a continuous sampling plan (CSP) for checking quality of personal computers adhering to the procedure described below.

In the beginning, 100 % of products are inspected. If i consecutive products are found to be free of defects, 100 % inspection will be discontinued and only a fraction (f) of the units will be inspected. These samples will be selected one at a time at random from the flow of production. If a sample is found to be defective, 100% inspection will be resumed.

Derive the probability that a product will be passed under this sampling inspection procedure. [10]