2016

BOOKLET NO. TESTCODE : PHB

Afternoon

Questions : 6+8+8	**Time : 2 hours**

On the answer-booklet write your Registration Number, Test Code, Number of this booklet, etc. in the appropriate places.

ATTENTION !

Please read the following very carefully before answering the test.

There are three parts. Part I is *compulsory* for all candidates and carries a credit of 30 % of the total. Besides, each candidate has to choose *only one* of the Parts II & III and answer from that part as per instructions. The credit for each of these parts is 70 % of the total.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER-BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP ! WAIT FOR THE SIGNAL TO START

Part I

Mathematical and Logical Reasoning

Answer all questions. All questions carry equal marks.

1. For any real x, let $f(x) = min\{\sqrt{x}, x^2\}$. Compute $\int_0^3 f(x)\,dx$.

2. A ray of light traveling along the line $y = 2$, in the direction of the positive x-axis, falls on a convex mirror in the form of a parabola. The equation of the parabola is $y^2 = 4x$. Find the equation of the line along which it will be reflected.

3. Find all possible real functions $f(x)$ such that, for $x \geq 0$

$$f(x) = \int_0^x f(t)\,dt.$$

4. A particle describes a curve $r = ae^{k\theta}$ under a central force P, where a and k are positive constants. Show that the force law will be the following:

$$P \propto \frac{1}{r^3}$$

5. At time $t = 0$ an object having mass m is released from rest at a height y_0 above the ground. Let g represent the constant gravitational acceleration. Derive an expression for the impact time (the time at which the object strikes the ground). What is the velocity with which the object strikes the ground?

6. Consider the vector function

$$\mathbf{v} = r^2\cos(\theta)\hat{\mathrm{r}} + r^2\cos(\phi)\hat{\theta} - r^2\cos(\theta)\sin(\phi)\hat{\phi}$$

   Evaluate the integral of the divergence of $\mathbf{v}$ over a volume one octant of the sphere of radius R.

Part II

Mathematics

Answer any five questions.

1. (a) Show that the escape velocity V for a satellite of the earth moving under the central force $\frac{\mu}{r^2}$ (r being the radial distance) per unit mass is given by $V = \sqrt{\frac{2\mu}{R}}$, where R is the radius of the earth and μ is a constant.

   (b) A solid circular cylinder of radius a rotating about its axis is placed gently with its axis horizontal on a rough plane, whose inclination to the horizon is α. Initially, the friction acts up the plane and the coefficient of friction is μ. Show that the cylinder will move upwards if $\mu > \tan\alpha$. Also show the time that elapses before rolling commence is

$$\frac{a\,\omega}{g(3\mu\cos\alpha - \sin\alpha)},$$

   where ω is the initial angular velocity of the cylinder.

[7+7]

2. (a) Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z+1)^4}dz$ where C stands for the circle $|z| = 3$.

   (b) Given that $u(x,y) = e^{-x}(x\ \sin y - y\cos y)$, find an analytic function $f(z)$ such that $f(z) = u(x,y) + iv(x,y)$.

[7+7]

3. (a) Show that

$$(1-x^2)P_n'(x) = xP_{n-1}(x) - nxP_n(x)$$

   where $P_n(x)$ denotes nth order Legendre Polynomial and the symbol $'$ indicates derivative with respect to x.

   (b) Show that the general solution of the equation $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$ can be written as $\phi(x,y) = f_1(x+iy) + f_2(x-iy)$ where $f_i(x,y)$ $(i = 1, 2)$ are twice differentiable arbitrary functions.

[8+6]

4. (a) Use Fourier transform to solve the equation

$$\frac{\partial u(x,t)}{\partial t} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = f(x), \quad |u(x,t)| < M,$$

where κ, M are constants, $t > 0$ and $-\infty < x < \infty$.

(b) Determine the nature of the equation

$$2\frac{\partial^2 u(x,y)}{\partial x^2} - 4\frac{\partial^2 u(x,y)}{\partial x \partial y} - 6\frac{\partial^2 u(x,y)}{\partial y^2} + \frac{\partial u(x,y)}{\partial x} = 0$$

Also determine its characteristic.

[7+7]

5. Let G_{25}^* denote the set of all integers between 1 and 25 which are coprime to 25. Define a binary operation $\odot$ on G_{25}^* as follows.

For $a, b \in G_{25}^*, a \odot b = c$ if $c \in G_{25}^*$ and $ab \equiv c \bmod 25$.

(a) Show that $(G_{25}^*, \odot)$ is a group.

(b) Hence, or otherwise, show that

$$13^{20} \equiv 1 \bmod 25.$$

[8+6]

6. (a) For $x \geq 0$, define $f(x) = \int\limits_0^x e^{-t^2} dt$.

Show that for $x > 0$, $f(x) > x\, e^{-x^2}$.

(b) For a real number x, let $[x]$ denote the largest integer less or equal to x. Find

$$\lim_{n \to \infty} \int\limits_1^{n+1} \left(\frac{t}{[t]} - 1 \right) dt.$$

[8+6]

7. (a) Let

$$f(x,y) = \begin{cases} e^{\frac{xy}{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Determine whether f is continuous at $(0,0)$.

(b) If A and B are two events such that $P(B) = 1$, then show that A and B are independent events.

[8+6]

8. (a) Find the inverse of the following 4×4 matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(b) Show that for any real or complex square matrix A, there is an α such that $\alpha I + A$ is non-singular, where I denotes the identity matrix.

[8+6]

Part III

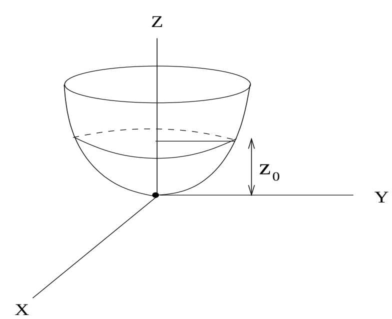
Physics

Answer any five questions.

1. (a) A particle moves without friction, on the inside of an axially symmetric vessel. The equation of the surface of the vessel is given by,

$$z = \frac{1}{2}b(x^2 + y^2)$$

where b is a constant and z represents the vertical direction, as shown in the figure. The particle is moving in a circular orbit at a constant height $z = z_0$. Obtain the particle's energy and angular momentum in terms of z_0, b, g, and m where g is the constant acceleration due to gravity and m is the mass of the particle.



(b) In an inertial frame, two events have the space time co-ordinates $\{x_1, y, z, t_1\}$ and $\{x_2, y, z, t_2\}$ where $x_2 - x_1 = 5c(t_2 - t_1)$. Consider another inertial frame which moves along x-axis with velocity u with respect to the first one. Find the value of u for which the events are simultaneous in the latter frame. (c represents the velocity of light in vacuum).

[(4+3)+7]

2. (a) Consider the Lagrangian,

$$L = e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right)$$

that represents a damped harmonic oscillator, where p is the momentum conjugate to q and m, ω, γ are constants.

i. Using the generating function $F_2(q, p, t) = e^{\gamma t} qP$, write down the Hamiltonian in Q, P co-ordinate system.

ii. Derive the equations of motion in this system.

(b) A particle of mass m moves in a plane in the field of force given by $\overrightarrow{F} = -\hat{r} kr \cos\theta$, where k is a constant and $\hat{r}$ is radial unit vector. Show that $mr^2\dot{\theta} =$ constant.

[(4+4)+6]

3. (a) A circular loop of wire of radius R, lies in the xy plane, centered at the origin, and carries a current I running counterclockwise as viewed from the positive z-axis.

i. What is its magnetic dipole moment?

ii. Evaluate the (approximate) magnetic field at points far from the origin.

iii. Show that for points on the z-axis, your answer in part ii is consistent with the exact field when $z >> R$.

(b) Consider a system with charge and current density ρ and $\vec{j}$ in vacuum.

i. Write down the Maxwell's equation.

ii. If the signs of all the sources (charge and current density) are reversed what happens to the electric and magnetic field?

iii. If the space is inverted, ($\vec{X} \rightarrow \vec{X\prime} = -\vec{X}$), what happens to charge and current density and to electric and magnetic field?

[(2+3+4)+(1+2+2)]

4. (a) Consider a diatomic crystal where atoms of mass M_1 lie on one set of planes and atoms of mass M_2 lie on planes interleaved between those of the first set. Find the condition of crossing between optical and acoustic branches for any k (k being the wave vector) within 0 to (π/a).

(b) The energy levels of a rigid rotor are $\epsilon_i = Aj(j+1)$ where $j = 0, 1, 2....$, and A is a constant. The degeneracy of each level is $g_j = 2j + 1$.

i. Find the general expression for the partition function Z using Boltzmann statistics.

ii. Show that at high temperature it can be approximated by an integral.

iii. Evaluate the high temperature energy and heat capacity C_v.

[7+(2+2+3)]

5. (a) 20 litres of gas at atmospheric pressure is compressed isothermally to a volume of 1 litre and then allowed to expand adiabatically to 20 litres.

   i. Sketch the processes on PV diagram for a monoatomic and a diatomic gas.

   ii. For both the cases discuss whether the "net work" - is being done on the system or by the system?

   (b) The speed of longitudinal wave of small amplitude in an ideal gas is

$$v = \sqrt{\frac{dP}{d\rho}}$$

   where p is the ambient gas pressure and $\rho(= M/V)$ is the corresponding gas density. Find the speed v of sound in a gas for which the compressions and rarifactions are adiabatic. The symbols have their usual meanings.

[(3+4)+7]

6. (a) i. Show that $(\vec{\sigma}.\vec{a})(\vec{\sigma}.\vec{b}) = \vec{a}.\vec{b} + i\vec{\sigma}.(\vec{a} \times \vec{b})$ where $\vec{a}$ and $\vec{b}$ are two vectors in $\Re^3$, and σ_x, σ_y and σ_z are usual Pauli spin matrices.

   ii. Consider the Dirac Hamiltonian of a free particle

$$H = c\vec{\alpha}.\vec{p} + \beta mc^2.$$

   Show that $\left(\vec{L} + \frac{1}{2}\hbar\vec{\sigma}\right)$ commutes with the Hamiltonian H. Here $\vec{L}$ is the orbital angular momentum and other symbols have their usual meanings.

   (b) Find the magnetic moment, in units of Bohr magneton, of an atom in the state ${}^3P_2$. In how many sub-states will the state split, if the atom is put in a weak magnetic field? Draw the splitting diagram.

[(3+4)+(2+2+3)]

7. (a) A particle in the infinite square well of width a has the initial wave function

$$\psi(x,0) = A\sin^3(\pi x/a), \qquad 0 \le x \le a$$

i. Find the normalization constant A.

ii. Determine the wave function $\psi(x,t)$ at a later time t.

iii. Calculate the expectation value of x as a function of time.

(b) The wave function of the harmonic oscillator at $t = 0$ is given by

$$\psi(0) = N\left(|0\rangle + 2|1\rangle\right)$$

where N is the normalization constant and $|n\rangle$ is the eigenfunction of corresponding energy eigenvalue $E_n = \hbar\omega\left(n + \dfrac{1}{2}\right)$. Calculate the minimum time in which the wave function evolves to its orthogonal state.

[(3+2+5)+4]

8. (a) Consider a complex scalar field ϕ having mass m.

i. Find the equations of motion.

ii. Find the Noether charge current.

iii. Now introduce a self-interaction term that is of fourth power in the complex scalar field and find the Noether charge current in this case also.

(b) Give reasons for why the reactions below are not allowed (the symbols have their usual meanings). Attempt any four.

i. $p \to e^+ + \pi^0$

ii. $p + n \to p + \Lambda^0$

iii. $K^- \to \pi^0 + e^-$

iv. $K^+ \to \pi^+ + \pi^+ + \pi^0$

v. $n \to p + \pi^-$

vi. $p \to e^+ + n + \nu_e$

vii. $p + \bar{p} \to \gamma$

[(2+2+2)+(2+2+2+2)]