2014

BOOKLET NO. TESTCODE : PHB

Afternoon

Questions : 6+8+8	**Time : 2 hours**

On the answer-booklet write your Registration Number, Test Code, Number of this booklet, etc. in the appropriate places.

ATTENTION !

Please read the following very carefully before answering the test.

There are three parts. Part I is *compulsory* for all candidates and carries a credit of 30% of the total. Besides, each candidate has to choose *only one* of the Parts II & III and answer from that part as per instructions. The credit for each of these parts is 70% of the total.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER-BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP ! WAIT FOR THE SIGNAL TO START

Part I

Mathematical and Logical Reasoning

Answer all questions. All questions carry equal marks.

1. Let
$$f(x) = \frac{xe^{1/x} - x}{e^{1/x}}; \quad x \in \mathbb{R}.$$
Find $\lim_{x \to \infty} f(x)$.

2. Let
$$A = \frac{1}{2} \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$
Find the eigenvalue of A^{2014}.

3. There are three balls labeled $1, 2, 3$ and three boxes also labeled $1, 2, 3$. Balls are placed at random into the boxes. Let X be the random variable that denotes the number of empty boxes. Find $E(X)$, the expectation of X.

4. A ball of unit mass is dropped from a height h. The frictional force of air is proportional to the velocity of the ball (with a constant of proportionality α). Show that the height of the ball in time t is
$$y(t) = h - \frac{g}{\alpha}\left[t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right],$$
where g is the acceleration due to gravity.

5. A satellite of mass m is in a stationary orbit above a point on the equator of the earth. The mass and radius of the earth are M and R respectively and its angular velocity about its own axis is ω. Find the height of the orbit above the ground.

6. A particle is constrained to move along the x-axis under the influence of the net force $F = -kx$ with amplitude A and frequency f, where k is a positive constant. When $x = \frac{A}{2}$, what is the speed of the particle?

Part II

Mathematics

Answer any five questions.

1. (a) A train with proper length L moves with speed $5c/13$ with respect to the ground, where c is the speed of light in vacuum. A ball is thrown from the back of the train towards the front. The speed of the ball with respect to the train is $c/3$. As viewed by someone from the ground, how much time does the ball spend in air and how much distance does it travel?

   (b) A particle is moving under the influence of a central force such that $r \propto \frac{1}{\theta}$. Determine the potential energy as a function of r. The symbols have their usual meanings.

[7+7]

2. (a) Find the residue of the function $f(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.

   (b) Using a suitable contour evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

[7+7]

3. (a) Use the generating function for the Hermite polynomials to find $H_0(x), H_1(x)$ and $H_2(x)$.

   (b) Expand x^3 in a series of Hermite polynomials.

[7+7]

4. (a) Consider a concave mirror in the shape of a parabola with *focus* F whose equation is given by $y^2 = 4x$. Let P be a point source of light *inside* the parabola. Find Q on the parabola such that the ray PQ on reflection passes through the focus F.

   (b) Let $\phi : (\mathbf{Q}, +) \to (\mathbf{Q}, +)$ be a homomorphism of the additive group of rationals into itself. Show that for some $\lambda \in \mathbf{Q}$

$$\phi(x) = \lambda x, \quad \text{for all } x \in \mathbf{Q}.$$

[7+7]

5. (a) Consider the following upper-triangular matrix A over Z_5, the field of integers modulo 5:

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Show that A is invertible and find its inverse over Z_5.

(b) Consider the linear space $\mathcal{M}$ of all $n \times n$ matrices over $I\!R$. Show that $\mathcal{M}$ has a basis consisting of skew-symmetric and symmetric matrices.

(c) Can there be a linear transformation $T : I\!R^3 \to I\!R^3$ such that T maps $(1, 0, 1), (0, -1, 0), (1, 1, 1)$ to $(1, 0, 0), (0, 1, 0), (0, 1, 1)$ respectively? Justify.

[6+4+4]

6. (a) Find the Fourier coefficients corresponding to the function (period = 10)

$$\begin{aligned} f(x) &= 0, \quad -5 < x < 0 \\ &= 3, \quad \;\; 0 < x < 5 \end{aligned}$$

Also write the Fourier series corresponding to this function.

(b) Let $p(x) = x^{2013} - x - 1$ be a real polynomial. Let $g(x)$ be a real-valued bounded continuous function. Show that there exists an $x_0 \in I\!R$ such that

$$p(x_0) = g(x_0).$$

[7+7]

7. (a) Let G be a group with no proper subgroup. Show that G is finite and hence cyclic.

(b) Let G be the group of all 2×2 non-singular real matrices with matrix multiplication as the group operation. Give an example of a non-trivial normal subgroup of G.

[7+7]

8. (a) Determine the regions where the equation

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + 3y^2\frac{\partial u}{\partial x} = 0$$

is parabolic, hyperbolic and elliptic.

(b) Find a recurrence formula and indicial equation for an infinite series solution around $x = 0$ for the differential equation

$$8x^2y'' + 10xy' + (x - 1)y = 0.$$

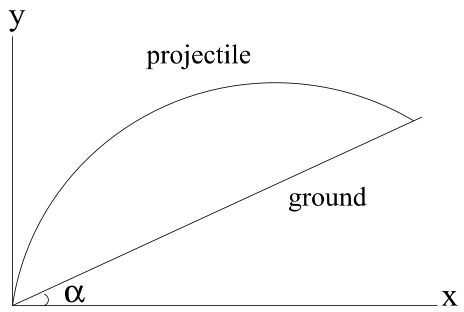
[7+7]

Part III

Physics

Answer any five questions.

1. (a) A projectile is fired uphill over the ground which slopes at an angle α to the horizontal (as shown in the figure). Find the direction in which it should be aimed to achieve the maximum range. (Hint: Use a relation between x and y coordinates where the projectile touches the ground.)



(b) Consider the following transformation from a canonical set of phase space coordinates $\{q, p\}$ to another set of phase space coordinates $\{Q, P\}$,

$$Q = q^{\alpha} \cos(\beta p), \quad P = q^{\alpha} \sin(\beta p)$$

where α and β are constants. Show that the transformation is canonical for $\alpha = \frac{1}{2}$ and $\beta = 2$.

[7+7]

2. (a) Consider the Lagrangian

$$L = \frac{1}{2} m(\dot{x}^2 - w^2 x^2) e^{\gamma t}$$

for the motion of a particle of mass m in one dimension. The constants m, γ and w are real and positive.
i) Derive the equation of motion.
ii) Find the canonical momentum, and from this construct the Hamiltonian function.

(b) A person standing at the rear of a railroad car fires a bullet towards the front of the car. The speed of the bullet, as measured in the frame of the car, is 0.5c (where c is the speed of light in vacuum) and the proper length of the car is 400m. The train is moving at 0.6c as measured by observers in the ground. For the ground observers, find
i) the length of the railroad car,
ii) the speed of the bullet,
iii) the time required for the bullet to reach the front of the car.

[(3+4)+(2+2+3)]

3. (a) Consider a simple harmonic oscillator in one dimension with the Hamiltonian

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

where a and $a^\dagger$ are the annihilation and creation operators respectively and the other symbols have their usual meanings. The ket vector of the harmonic oscillator at $t = 0$ is given by

$$|\psi(0)\rangle = N\left(|0\rangle + 2|1\rangle + 3|2\rangle\right)$$

where N is the normalization constant and $|n\rangle$ is the eigenket of corresponding energy eigenvalue $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$.
i) Find the normalization constant N.
ii) Calculate the probability of finding the energy to be $\frac{3}{2}\hbar\omega$ on energy measurement.
iii) Find the ket vector $|\psi(t)\rangle$ at time t and calculate expectation value of the energy for this ket vector.

(b) Consider the following ket vector of the harmonic oscillator

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\mu|^2} \frac{\mu^n}{\sqrt{n!}} |n\rangle$$

where $|n\rangle$ is an eigenket of the Hamiltonian with eigenvalue $\left(n + \frac{1}{2}\right)\hbar\omega$ and μ is a complex number. Show that

$$\hat{a}|\phi\rangle = \mu|\phi\rangle$$

where $\hat{a}$ is the harmonic oscillator annihilation operator.

[(1+2+3)+8]

4. (a) Consider N fixed non-interacting magnetic moments each of magnitude μ_0. The system is in thermal equilibrium at temperature T and is in a uniform external magnetic field B. Each magnetic moment can be oriented only parallel or antiparallel to B. Calculate
i) the partition function,
ii) the specific heat.

(b) For a degenerate, spin $\frac{1}{2}$, non-interacting Fermi gas at zero temperature, find an expression for the energy of a system of N such particles confined to a volume V. Assume the particles are non-relativistic.

[(5+5)+4]

5. (a) A sphere of radius R_1 has charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located at a distance a from the centre.
i) Find the field $\mathbf{E}$ at the centre of the hollow sphere.
ii) Find the potential ϕ at the same point.

(b) An electric charge Q is uniformly distributed over the surface of a sphere of radius r. Show that the force on a small charge element dq is radial and outward and is given by

$$d\mathbf{F} = \frac{1}{2}\mathbf{E}dq$$

where $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$ is the electric field at the surface of the sphere. ε_0 is the permittivity of the free space.

[(4+4)+6]

6. (a) A particle moves in a time-independent electric field $\mathbf{E} = -\nabla\phi$ and any magnetic field $\mathbf{B}$. Using Lorentz force law, can you show that the energy of the particle is constant?

(b) A particle moves along the x-axis in the electric field $\mathbf{E} = A\exp^{-t/\tau}\,\hat{\mathbf{i}}$ (where A and τ are constants) and the magnetic field is zero along the x-axis. Find $x(t)$ with the initial conditions $x(0) = \dot{x}(0) = 0$.

(c) A one-dimensional quantum harmonic oscillator (with ground state energy $\hbar\omega/2$) is in thermal equilibrium with a heat bath at temperature T. Determine the mean value of the oscillator's energy, $\langle E \rangle$, as a function of T.

[4+4+6]

7. (a) A particle moves in a central potential $V(r) = -\frac{g^2}{r^{3/2}}$, where g is a constant. Using the normalized ground state wave function of the Hydrogenic atom $\psi(r) = \left(\frac{k^3}{8\pi}\right)^{1/2} e^{-kr/2}$, find out the upper bound of the lowest s-state energy.

   (b) Consider a relativistic field theory involving two scalar fields ϕ and ψ with ϕ having mass parameter m and ψ being massless. The interaction term is $g\phi^2\psi^2$, where g is the coupling constant.
   i) Write down the action of the system in two space and one time dimensions.
   ii) Derive the equations of motion. Write down the effective masses for ψ and ϕ.
   iii) Consider the process where one ϕ particle and one ψ particle of momenta p_1^μ and p_2^μ respectively scatters to one ϕ particle and one ψ particle of momenta q_1^μ and q_2^μ respectively. Draw Feynman diagrams for the lowest order and next to lowest order processes with proper labeling of the momenta.

[7+(1+3+3)]

8. (a) Explain why the following processes are not observed in nature. Discuss any four of the seven options. (The symbols carry their usual meaning.)

$$
\begin{aligned}
\mathrm{p} &\rightarrow \mathrm{e}^+ + \pi^0 \\
\Lambda^0 &\rightarrow \mathrm{K}^0 + \pi^0 \\
\mathrm{p} + \bar{\mathrm{p}} &\rightarrow \Lambda^0 + \Lambda^0 \\
\Lambda^0 &\rightarrow \mathrm{K}^+ + \mathrm{K}^- \\
\mathrm{n} &\rightarrow \mathrm{p} + \mathrm{e}^- \\
\mathrm{p} &\rightarrow \mathrm{e}^+ + \nu_\mathrm{e} \\
\mu^+ &\rightarrow \mathrm{e}^+ + \gamma
\end{aligned}
$$

(b) Very high energy protons in cosmic rays can lose energy through a collision process

$$p + \gamma \rightarrow p + \pi.$$

The typical energy radiated in this process is 2.73 K. How energetic need a cosmic ray proton be to be above the threshold for this reaction? Given that the Boltzmann constant $k = 8.6 \times 10^{-5}$ eV/K, and the masses of proton and pion are 0.938 GeV and 0.140 GeV respectively.

[(4 × 2)+6]