Test Code : CSB (Short Answer Type) 2017

Junior Research Fellowship (*JRF) in Computer Science*

The CSB test booklet will have two groups as follows:

GROUP A

A test of aptitude for Computer Science for all candidates in the basics of computer programming and mathematics, as indicated in the syllabus.

GROUP B

A test, divided into five sections in the following areas at M.Sc./M.E./M.Tech. level:

- Mathematics,
- Statistics,
- Physics,
- Electrical and Electronics Engineering, and
- Computer Science.

A candidate has to answer questions from **only** one of these sections in GROUP B, according to his/her choice.

Group A carries **40 marks** *and Group B carries* **60 marks**.

The syllabi and sample questions of the CSB test are given overleaf.

Sample Questions

Note that all questions in the sample set are not of same marks and same level of difficulty.

GROUP A

A1. The king's minter keeps mn coins in n boxes each containing m coins. Each box contains 2 false coins out of m coins. The king suspects the minter and randomly draws 1 coin from each of the n boxes and has these tested. What is the probability that the minter's dishonest actions go undetected?

A2. Consider the pseudo-code given below.
Input: Integers b and c.

1. $a_0 \leftarrow \max(b, c)$, $a_1 \leftarrow \min(b, c)$.
2. $i \leftarrow 1$.
3. Divide a_{i-1} by a_i . Let q_i be the quotient and r_i be the remainder.
4. If $r_i = 0$ then go to Step 8.
5. $a_{i+1} \leftarrow a_{i-1} - q_i * a_i$.
6. $i \leftarrow i + 1$.
7. Go to Step 3.
8. Print a_i .

What is the output of the above algorithm when $b = 28$ and $c = 20$? What is the mathematical relation between the output a_i and the two inputs b and c?

A3. Consider the sequence $a_n = a_{n-1}\ a_{n-2} + n$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 1$. Is a_{2011} odd? Justify your answer.

A4. Given an array of n integers, write pseudo-code for reversing the contents of the array without using another array. For example, for the array
10 15 3 30 3
the output should be
3 30 3 15 10.
You may use one temporary variable.

A5. The integers 1, 2, 3, 4 and 5 are to be inserted into an empty stack using the following sequence of PUSH() operations:

PUSH(1) PUSH(2) PUSH(3) PUSH(4) PUSH(5)

Assume that POP() removes an element from the stack and outputs the same. Which of the following output sequences can be generated by inserting suitable POP() operations into the above sequence of PUSH() operations? Justify your answer.

(a) 5 4 3 2 1
(b) 1 2 3 4 5
(c) 3 2 1 4 5
(d) 5 4 1 2 3.

A6. Derive an expression for the maximum number of regions that can be formed within a circle by drawing n chords.

A7. Given $A = \{1, 2, 3, \ldots, 70\}$, show that for any six elements a_1, a_2, a_3, a_4, a_5 and a_6 belonging to A, there exists one pair a_i and a_j for which $|a_i - a_j| \leq 14$ $(i \neq j)$.

A8. The function `RAND()` returns a positive integer from the uniform distribution lying between 1 and 100 (including both 1 and 100). Write an algorithm (in pseudo-code) using the given function `RAND()` to return a number from the binomial distribution with parameters $(100, 1/4)$.

A9. Calculate how many integers in the set $\{1, 2, 3, \ldots, 1000\}$ are not divisible by 2, 5, or 11.

A10. Let M be a 4-digit positive integer. Let N be the 4-digit integer obtained by writing the digits of M in reverse order. If $N = 4M$, then find M. Justify your answer.

A11. Consider all the permutations of the digits $1, 2, \ldots, 9$. Find the number of permutations each of which satisfies *all* of the following:

- the sum of the digits lying between 1 and 2 (including 1 and 2) is 12,
- the sum of the digits lying between 2 and 3 (including 2 and 3) is 23,

- the sum of the digits lying between 3 and 4 (including 3 and 4) is 34, and
- the sum of the digits lying between 4 and 5 (including 4 and 5) is 45.

A12. State, with justification, which of the following expressions f, g and h, define valid real-valued functions over the set of positive rational numbers. We denote a rational number by m/n, where m and n are positive integers.

(a) $f(m/n) = 2^m - 2^n$.

(b) $g(m/n) = \log m - \log n$.

(c) $h(m/n) = (m^2 - n^2)/(mn)$.

A13. Given a function $f : A \to A$, an element $x \in A$ is said to be a fixed point of f if and only if $f(x) = x$. Let $f : \{1, 2, \ldots, 100\} \to \{1, 2, \ldots, 100\}$ be a function. For all $S \subseteq \{1, 2, \ldots, 100\}$, suppose a procedure `FIXED`(S) can determine whether the function f has at least one fixed point in S or not. Define a strategy to determine whether the function f has at least two fixed points by executing the procedure `FIXED` at most 15 times.

A14. There are n students in a class. The students have formed k committees. Each committee consists of more than half of the students. Show that there is at least one student who is a member of more than half of the committees.

A15. Let $D = \{d_1, d_2, \ldots, d_k\}$ be the set of distinct divisors of a positive integer n (D includes 1 and n). Show that

$$\sum_{i=1}^{k} \sin^{-1} \sqrt{\log_n d_i} = \frac{\pi}{4} \times k.$$

HINT: $\sin^{-1} x + \sin^{-1} \sqrt{1 - x^2} = \frac{\pi}{2}$

A16. Give a strategy to sort four distinct integers a, b, c, d in increasing order that minimizes the number of pairwise comparisons needed to sort any permutation of a, b, c, d.

A17. An $n \times n$ matrix is said to be *tridiagonal* if its entries a_{ij} are zero except when $|i - j| \leq 1$ for $1 \leq i, j \leq n$. Note that only $3n - 2$ entries of a tridiagonal matrix are non-zero. Thus, an array L of size $3n-2$ is sufficient to store a tridiagonal matrix. Given i, j, write pseudo-code to

(a) store a_{ij} in L, and

(b) get the value of a_{ij} stored in L.

A18. Consider an $m \times n$ integer grid. A *path* from the lower left corner at $(0,0)$ to the grid point (m,n) can use three kinds of steps, namely (i) $(p,q) \to (p+1,q)$ (horizontal), (ii) $(p,q) \to (p,q+1)$ (vertical), or (iii) $(p,q) \to (p+1,q+1)$ (diagonal). Derive an expression for $D_{m,n}$, the number of such distinct paths.

A19. The numbers $1, 2, \ldots, 10$ are circularly arranged. Show that there are always three adjacent numbers whose sum is at least 17, irrespective of the arrangement.

A20. Consider six distinct points in a plane. Let m and M denote respectively the minimum and the maximum distance between any pair of points. Show that $M/m \geq \sqrt{3}$.

A21. Consider the following intervals on the real line:

$$\begin{array}{ll} A_1 = (13.3, 18.3) & A_3 = (8.3, 23.3) - A_1 \cup A_2 \\ A_2 = (10.8, 20.8) - A_1 & A_4 = (5.8, 25.8) - A_1 \cup A_2 \cup A_3 \end{array}$$

where $(a,b) = \{x : a < x < b\}$.

Write pseudo-code that finds the interval to which a given input $x \in (5.8, 25.8)$ belongs, i.e., your pseudo-code should calculate $i \in \{1,2,3,4\}$ such that $x \in A_i$. Your method should not use any comparison operation.

A22. A group of 15 boys plucked a total of 100 apples. Prove that two of those boys plucked the same number of apples.

A23. How many 0's are there at the end of 50! ?

A24. Suppose X is a set such that for every function $f : X \to X$, f is one-to-one if and only if f is onto. Show that every one-to-one function $f : P(P(X)) \to P(P(X))$ is onto, where $P(A)$ denotes the set of all subsets of a set A.

A25. Given an array $A = \{a_1, a_2, \ldots, a_n\}$ of unsorted distinct integers, write a program in *pseudo-code* for the following problem: given an integer u, arrange the elements of the array A such that all the elements in A which are less than or equal to u are at the beginning of the array, and the elements which are greater than u are at the end of the array. You may use at most 5 extra variables apart from the array A.

A26. The vertices of a triangle T are given. For an arbitrary point P in the plane, give an algorithm to test if P belongs to the interior of T. (The interior of T does not include its edges).

A27. Find the value of $\sum ij$, where the summation is over all integers i and j such that $1 \leq i < j \leq 10$.

A28. Let $S = \{x \in \mathbb{R} : 1 \leq |x| \leq 100\}$. Find all subsets M of S such that for all x, y in M, their product xy is also in M.

A29. Let us consider the following 2-person game: the players alternately choose a number. The first player starts with a number between 1 to 10, and the players then pick up a number within the next ten of the number that his opponent has chosen earlier. The player who is able to select the number 100 first, wins the game. Can the first player pick up a number between 1 to 10 such that whatever may be the strategy of his opponent, the first player will be able to reach 100 first ?

GROUP B

(i) MATHEMATICS

M1. (a) Show that the sequence given by $x_n = \displaystyle\int_0^n \frac{\sin x}{x} dx, n \geq 1$ is Cauchy.

(b) Define the function on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \dfrac{\sin x \sin y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Is f continuous at $(0,0)$? Justify your answer.

M2. (a) Suppose f is a real-valued continuous function defined on $[0,1]$ such that $(2x-1)(2f(x)-x) > 0$ for all $x \neq 1/2$. If $f'(1/2)$ exists, then show that it cannot be smaller than $1/2$.

(b) Suppose f is a real-valued continuous function defined on $[-2,2]$ and is three times differentiable in $(-2,2)$. If $f(2) = -f(-2) = 4$ and $f'(0) = 0$, then show that there exists $x \in (-2,2)$ such that $f'''(x) \geq 3$.

(c) Find all values of $\alpha \in \mathbb{R}$ for which the sum

$$\sum_{n \geq 1} n^{n^\alpha} e^{-n}$$

is convergent.

M3. (a) Compute the following limit:

$$\lim_{n \to \infty} \frac{\sin 1 + 2\,\sin\frac{1}{2} + 3\,\sin\frac{1}{3} + \cdots + n\,\sin\frac{1}{n}}{n}.$$

(b) Show that there exists no one-to-one function $f : \mathbb{R} \to \mathbb{R}$ with the property that for all $x \in \mathbb{R}$, $f(x^2) - (f(x))^2 \geq 1/4$.

(c) If f is a real-valued continuous function defined on $[0,1]$ such that

$$\int_0^1 f(x)\, x^n \, dx = 0$$

for all $n = 0, 1, 2, \ldots$, then show that $f(x) = 0$ for all $x \in [0,1]$.

M4. (a) Suppose f is a continuous real valued function on $[0,1]$. Show that

$$\int_0^1 xf(x)\,dx = \frac{1}{2}f(\xi)$$

for some $\xi \in [0,1]$.

(b) For every $x \geq 0$, prove that

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+\theta(x)}},$$

for a unique $\theta(x), 0 < \theta(x) < 1$. Also prove that,
(i) $\frac{1}{4} \leq \theta(x) \leq \frac{1}{2}$, and
(ii) $\lim_{x\to 0}\theta(x) = \frac{1}{4}$ and $\lim_{x\to\infty}\theta(x) = \frac{1}{2}$.

M5. (a) If p is a prime ($p \neq 2$) and $p \mid (m^p+n^p)$, (i.e., p divides (m^p+n^p)) then show that $p^2 \mid (m^p+n^p)$.

(b) Let f be a continuous function on $[0,1]$. Suppose for each integer $n \geq 1$,

$$\int_0^1 f^3(x)x^n dx = 0, \quad \text{where} \quad f^3(x) = (f(x))^3.$$

Show that $\int_0^1 f^4(x)dx = 0$. Hence, or otherwise, show that $f \equiv 0$.

M6. Let S_n be the group of all permutations of $\{1,\ldots,n\}$ under composition.

(a) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ be an element of S_6. Find the order of the cyclic subgroup generated by σ.

(b) Find the minimum n such that S_n contains a cyclic subgroup of order 30. Justify your answer.

(c) (i) Let S be a cyclic group of order 6. Show that S has a unique subgroup of order 3.

(ii) Let S be a finite cyclic group and K be a subgroup of S of order m. Show that an element $a \in S$ is an element of K if and only if $a^m = e$.

M7. Let $G = (V,E)$ be a graph and $e \in E$ be an edge whose endpoints are v_1 and v_2. Define the graph $G_e = (V',E')$ formed by merging v_1 and v_2 into a single vertex v as follows:

$$
\begin{aligned}
V' = \quad & (V \setminus \{v_1, v_2\}) \cup \{v\}, \\
E' = \quad & \Big\{E \setminus (\{e\} \cup \{(u, v_1) \in E\} \cup \{(u, v_2) \in E\})\Big\} \\
& \cup \Big\{(u, v) : (u, v_1) \in E \text{ or } (u, v_2) \in E\Big\}.
\end{aligned}
$$

Note, if there is a vertex $u \in V$ such that both (u, v_1) and (u, v_2) are in E, then the edge (u, v) occurs twice in G_e.

(a) If G is Eulerian, then is G_e Eulerian for all edges e of G? If G is not Eulerian, then is it possible for G_e to be Eulerian for some edge e of G? Justify your answer.

(b) Give an example of a bipartite graph G on at least 8 vertices and an edge e of G such that G_e is also bipartite.

(c) Give an example of a graph G on at least 6 vertices and an edge e in G such that the chromatic number of G_e is one less than the chromatic number of G.

M8. For variables x_1 and x_2, and $\alpha = (a_1, a_2)$, let the monomial $x_1^{a_1} x_2^{a_2}$ be denoted by x^α. Let A be a non-empty subset of pairs $\alpha = (a_1, a_2)$, where a_1 and a_2 are non-negative integers. For a field $\mathbb{F}$, define $I(A) = \{h_1 x^{\alpha_1} + \cdots + h_k x^{\alpha_k} : k \geq 0, \alpha_1, \ldots, \alpha_k \in A, h_1, \ldots, h_k \in \mathbb{F}[x_1, x_2]\}$.

(a) Show that $I(A)$ is an ideal of $\mathbb{F}[x_1, x_2]$.

(b) Suppose $g(x_1, x_2) = \sum_{i=1}^{\ell} g_i x^{\beta_i}$ with $g_i \in \mathbb{F}$ is in $I(A)$. Show that for $1 \leq i \leq \ell$, each x^{β_i} is divisible by some x^{α_i} with $\alpha_i \in A$.

(c) Let $A = \{(1, i), (j, 1) : i, j \geq 2\}$. Find a set of pairs S of minimum possible cardinality such that $I(A) = I(S)$.

(d) If I is an ideal of $\mathbb{F}[x_1, x_2]$, define

$$\sqrt{I} = \{f : f^m \in I \text{ for some } m \geq 0\}.$$

Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.

M9. (a) Show that the following function of two $n \times n$ matrices A and B with real entries defines an inner product of matrices.

$$\langle A, B \rangle = \operatorname{trace}(\mathrm{A^T B}),$$

where A^T is the transpose of A.

(b) The norm of a matrix A is defined by

$$\| A \| = \langle A, A \rangle^{\frac{1}{2}}.$$

(i) Show that $\| AB \| \leq \| A \| \| B \|$.

(ii) For a sequence of matrices $\{A_k\}$ and another matrix A, we define A_k converges to A if $\| A_k - A \| \to 0$ as $k \to \infty$. Show that if for a matrix A, $\{A^k\}$ converges to B, then $B^2 = B$.

(c) Let A and B be two $n \times n$ symmetric matrices such that $AB = BA$. Show that if $x \neq 0$ is an eigenvector of A and $Bx \neq 0$, then Bx is also an eigenvector of A corresponding to the same eigenvalue.

M10. (a) Let $\{a_n\}_{n \geq 0}$ be a non-negative sequence satisfying $a_{m+n} < a_m + a_n + 1$, for all positive integers m, n. Show that

(i) $a_{17} < 2a_8 + a_1 + 2$.

(ii) $\frac{a_n}{n}$ converges as $n \to \infty$.

(b) Let ε_n be the fractional part of $n!e$, where n is a positive integer.

(i) Show that $\frac{1}{n+1} < \varepsilon_n < \frac{1}{n}$ for all positive integers n.

(ii) Prove that $n \sin(2n!e\pi)$ converges to 2π as $n \to \infty$.

M11. (a) Find the remainder when 20^{13} is divided by 4940.

(b) Let R be a ring and I, J be ideals of R. Define IJ as the set of all elements that can be written as finite sums of elements of the form xy, where $x \in I$ and $y \in J$.

i. Is IJ an ideal of R? Justify your answer.

ii. Which is a bigger set: IJ or $I \cap J$? Prove your statement.

M12. (a) Find a basis for the following subspace of $\mathbb{R}^4$:

$$\{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + 3x_4 = 0, x_1 + x_2 - 2x_3 + x_4 = 0\}.$$

(b) Is it possible to compute the dimension of the above subspace without explicitly finding a basis? If yes, how? If not, justify your answer.

(c) For all $k \in \mathbb{Z}$, define $y_k = \displaystyle\int_{-1}^{1} x^2 e^{ikx} dx$. Show that for any positive integer n, $((a_{r,s} = y_{r-s}))_{1 \leq r,s \leq n}$ is a positive semi-definite matrix.

M13. (a) Find the minimum value of the determinant

$$\begin{vmatrix} 1 & \rho & x \\ \rho & 1 & y \\ -x & -y & 1 \end{vmatrix},$$

where $|\rho| < 1$ is a constant.

(b) Suppose $a_1, a_2, a_3, b_1, b_2, b_3$ are six real numbers. Define the matrix E as

$$E = \begin{pmatrix} a_1^2 + b_1^2 & a_1a_2 + b_1b_2 & a_1a_3 + b_1b_3 \\ a_1a_2 + b_1b_2 & a_2^2 + b_2^2 & a_2a_3 + b_2b_3 \\ a_1a_3 + b_1b_3 & a_2a_3 + b_2b_3 & a_3^2 + b_3^2 \end{pmatrix}.$$

Compute the eigen-value of E that has the smallest absolute value.

M14. (a) Show that in a connected graph, any two longest paths have at least one vertex in common.

(b) Construct a cubic graph with $2n$ vertices having no triangles. (A graph is cubic if every vertex has degree three.)

(c) Let G be a graph on 9 vertices. Show that either G has a triangle or $\overline{G}$ contains K_4. (Here $\overline{G}$ is the complement of the graph G, and K_4 is the complete graph with 4 vertices). [Hint: any vertex has degree at least 4 in either G or $\overline{G}$.]

(d) Let $S = \{0, 1, 2, 3\}$ and let a_k be the number of strings of length k over S having an even number of zeros. Find a recurrence relation for a_k and then solve for a_k.

M15. (a) Let T be a tree with n vertices ($n \geq 3$). For any positive integer i, let p_i denote the number of vertices of degree i. Prove that

$$p_1 - p_3 - 2p_4 - \cdots - (n-3)p_{n-1} = 2.$$

(b) Show that 2^n does not divide $n!$ for any $n \geq 1$.

M16. (a) Show that, given $2^n + 1$ points with integer coordinates in $\mathbb{R}^n$, there exists a pair of points among them such that all the coordinates of the midpoint of the line segment joining them are integers.

(b) Find the number of functions from the set $\{1, 2, 3, 4, 5\}$ onto the set $\{1, 2, 3\}$.

M17. (a) A set S contains integers 1 and 2, and all integers of the form $3x + y$ where x and y are distinct elements of S. What is S ? Justify your answer.

(b) Let $\phi(n)$ denote the number of positive integers m relatively prime to $n; m < n$. Let $n = pq$ where p and q are prime numbers. Then show that $\phi(n) = (p-1)(q-1) = pq(1 - \frac{1}{q})(1 - \frac{1}{p})$.

M18. Consider the $n \times n$ matrix $A = ((a_{ij}))$ with $a_{ij} = 1$ for $i < j$ and $a_{ij} = 0$ for $i \geq j$. Let

$$V = \{f(A) : f \text{ is a polynomial with real coefficients}\}.$$

Note that V is a vector space with usual operations. Find the dimension of V, when (a) $n = 3$, and (b) $n = 4$. Justify your answer.

M19. Consider the following system of equations over a field $\mathbf{F}$.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2, \end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbf{F}$. State the conditions for which the above system of equations has (a) no solution, (b) a unique solution and (c) more than one solution.

Next consider the following equations over the field $\mathbf{Z}_p$, the field of integers modulo p, where p is a prime.

$$\begin{aligned} 7x - y &= 1 \\ 11x + 7y &= 3 \end{aligned}$$

For what value of p does this system of equations have (a) no solution, (b) a unique solution and (c) more than one solution? Find the solution in the case of (b).

M20. In each of the following four problems, two sets S and T are given. You need to define a continuous and onto function $f : S \to T$ in each of the cases. You need to provide mathematical reasons if you cannot define such an f.

(a) $S = (0,1)$, $T = (0,1]$.

(b) $S = (0,1]$, $T = (0,1)$.

(c) $S = (0,1) \cup (2,3)$, $T = (0,1)$.

(d) $S = (0,1)$, $T = (0,1) \cup (2,3)$.

M21. You are given 49 balls of colour red, black and white. It is known that, for any 5 balls of the same colour, there exist at least two among them possessing the same weight. The 49 balls are distributed in two boxes. Prove that there are at least 3 balls which lie in the same box possessing the same colour and having the same weight.

M22. Consider a sequence $\{a_n\}$ such that $0 < a_1 < a_2$ and $a_i = \sqrt{a_{i-1}a_{i-2}}$ for $i \geq 3$.

(a) Show that $\{a_{2n-1} : n = 1, 2, \ldots\}$ is an increasing sequence and $\{a_{2n} : n = 1, 2, \ldots\}$ is a decreasing sequence.

(b) Show that $\lim\limits_{n\to\infty} a_n$ exists.

(c) Find $\lim\limits_{n\to\infty} a_n$.

M23. (a) Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} z^{n^4}$.

(b) Test whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{inx}}{n^2}$ is uniformly convergent on $\mathbb{R}$.

(c) Consider all triangles on the plane with fixed perimeter $c \in \mathbb{R}$. Find the triangle whose area is maximum.

M24. If every vertex of a graph $G = (V, E)$ has degree at least $\frac{|V|}{2}$, then prove that there is a simple cycle containing all the vertices. Note that, a simple cycle with a vertex set $U \subseteq V$ is a cycle where each vertex of U appears only once if one traverses along the cycle.

M25. (a) Consider a graph with 8 vertices. If the degrees of seven of the vertices are $1, 2, 3, 4, 5, 6$ and 7, find the degree of the eighth vertex. Argue whether the graph is planar. Also find its chromatic number.

(b) Let $\mathbf{S}$ be a subset of $\{10, 11, 12 \ldots, 98, 99\}$ containing 10 elements. Show that there will always exist two disjoint subsets $\mathbf{A}$ and $\mathbf{B}$ of $\mathbf{S}$ such that the sum of the elements of $\mathbf{A}$ is the same as that of $\mathbf{B}$.

M26. (a) Determine the product of all distinct positive integer divisors of 630^4.

(b) Let $p_1 < p_2 < \ldots < p_{31}$ be prime numbers such that 30 divides $p_1^4 + p_2^4 + \cdots + p_{31}^4$. Prove that $p_1 = 2$, $p_2 = 3$ and $p_3 = 5$.

(c) Find all primes p and q such that $p + q = (p - q)^3$. Justify your answer.

M27. (a) Show that each of the equations $\sin(\cos x) = x$ and $\cos(\sin y) = y$ has exactly one root in $[0, \pi/2]$. If x_1 and x_2 are the roots of these two equations respectively, then show that $x_1 < x_2$.

(b) Let f be a real-valued continuous function on $\mathbb{R}$ satisfying the inequality

$$f(x) \leq \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy, \quad \forall x \in \mathbb{R},\ \forall h > 0.$$

Prove that for any bounded closed interval, the maximum of f on that interval is attained at one of its end points.

(c) Define $f(x) = e^{x^2/2} \int_x^{\infty} e^{-t^2/2}\, dt$ for $x > 0$. Show that $0 < f(x) < 1/x$ and $f(x)$ is monotonically decreasing for $x > 0$.

M28. (a) For $n \in \mathbb{N}$, let the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be given by

$$0 < b_1 < a_1, \quad a_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n}, \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

Show that both the sequences are monotone and they have the same limit.

(b) Consider a polynomial $f_n(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ with integer coefficients. If $f_n(b_1) = f_n(b_2) = f_n(b_3) = f_n(b_4) = f_n(b_5) = 19$ for five distinct integers b_1, b_2, b_3, b_4 and b_5, then how many different integer solutions exist for $f_n(x) = 23$? Justify your answer.

M29. (a) Show that $f(x) = e^{|x|} - x^5 - x - 2$ has at least two real roots, where e is the base of natural logarithms.

(b) Let $\sum a_n$ be a convergent series such that $a_n \geq 0$ for all n. Show that $\sum \sqrt{a_n}/n^p$ converges for every $p > \frac{1}{2}$.

M30. (a) Let D be an integral domain and $a \in D$. Show that $X^2 - a$ is reducible in $D[X]$ if and only if a is a square in D.

(b) Deduce that if A is a unique factorisation domain (UFD) of characteristic zero, then $A[X, Y]/(X^2 + Y^2 - 1)$ is an integral domain.

(c) Suppose that B is a UFD of positive characteristic p. Give a necessary and sufficient condition on p for which $B[X, Y]/(X^2 + Y^2 - 1)$ will be an integral domain.

(d) Let $\mathbb{Z}$ denote the ring of integers and $R = \mathbb{Z}[X, Y]/(X^2 + Y^2 - 1)$. Give two examples each (with justifications) of (a) maximal ideals in R and (b) prime ideals in R which are not maximal.

(ii) STATISTICS

S1. (a) Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables satisfying $X_{n+1} = X_n + Z_n$ (addition is modulo 5), where $\{Z_n\}_{n\geq 1}$ is a sequence of independent and identically distributed random variables with common distribution
$P(Z_n = 0) = 1/2$, $P(Z_n = -1) = P(Z_n = +1) = 1/4$.
Assume that X_1 is a constant belonging to $\{0, 1, 2, 3, 4\}$. What happens to the distribution of X_n as $n \to \infty$?

(b) Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with a common uniform distribution on $\{1, 2, \ldots, m\}$. Define a sequence of random variables $\{X_n\}_{n\geq 1}$ as $X_{n+1} = MAX\{X_n, Y_n\}$ where X_1 is a constant belonging to $\{1, 2, \ldots, m\}$. Show that $\{X_n\}_{n\geq 1}$ is a Markov chain and classify its states.

S2. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the gamma distribution with density function

$$f(x, \theta) = \frac{\theta^k}{\Gamma(k)} e^{-\theta x} x^{k-1}, \quad 0 < x < \infty,$$

where $\theta > 0$ is unknown and $k > 0$ is known. Find a minimum variance unbiased estimator for $\frac{1}{\theta}$.

S3. (a) Let $0 < p < 1$ and $b > 0$. Toss a coin once where the probability of occurrence of head is p. If head appears, then n independent and identically distributed observations are generated from Uniform $(0, b)$ distribution. If the outcome is tail, then n independent and identically distributed observations are generated from Uniform $(2b, 3b)$ distribution. Suppose you are given these n observations $X_1, \ldots, X_n$, but not the outcome of the toss. Find the maximum likelihood estimator of b based on $X_1, \ldots, X_n$. What happens to the estimator as n goes to ∞?

(b) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed as $U(\theta, \theta + |\theta|)$, where $\theta \neq 0$. Find the MLE of θ based on $X_1, X_2, \ldots, X_n$.

S4. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common density function f. Define the random variable

N as

$$N = n, \text{ if } X_1 \geq X_2 \geq \cdots \geq X_{n-1} < X_n; \text{ for } n = 2, 3, 4, \ldots.$$

Find $Prob(N = n)$. Find the mean and variance of N.

S5. (a) Suppose $X_1, \ldots, X_n$ $(n > 2)$ are independent and identically distributed observations from a normal population with mean θ and variance 1, $\theta \in \mathbb{R}$. Let $S_n = X_1 + \cdots + X_n$. Consider the conditional expectation of $X_1 X_2$ given S_n. Decide, with adequate reasons, if this conditional expectation depends on θ. Find an expression for the conditional expectation.

(b) Suppose $X_1, \ldots, X_n$ $(n > 2)$ are independent observations from a Poisson population with mean θ, $\theta > 0$. Suppose we are interested in estimating $\psi(\theta) = P_\theta(X_1 = 0) = e^{-\theta}$. Let $S_n = X_1 + \cdots + X_n$. As $\mathrm{E}(S_n/n) = \theta$, an estimator of $\psi(\theta)$, obtained by the method of moments, is given by $T_n = \exp(-S_n/n)$. Find the mean squared error of T_n. Also, find the limit of the mean squared error as n tends to infinity.

S6. (a) Let X and Y be two random variables such that

$$\begin{pmatrix} \log X \\ \log Y \end{pmatrix} \sim N(\mu, \Sigma).$$

Find a formula for $\phi(t, r) = E(X^t Y^r)$, where t and r are real numbers, and E denotes the expectation.

(b) If $\mathbf{X} \sim N(\mu, \mathbf{\Sigma})$, describe how you will find the smallest region C (the region with minimum volume) such that $P(\mathbf{X} \in C) = 0.75$. Justify your answer.

S7. (a) Let $X_1, X_2, \ldots, X_n$ (n ≥ 3) be identically distributed random variables with $Var(X_1) = 1$ and $Cov(X_i, X_j) = \rho < 0$ for all $i \neq j$. For any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in R^n$ (the superscript T stands for transpose), define $S(\boldsymbol{\alpha}) = \sum_{i=1}^n \alpha_i X_i / \|\boldsymbol{\alpha}\|$.

(i) Find $\boldsymbol{\alpha}$ such that $V(\boldsymbol{\alpha}) = Var(S(\boldsymbol{\alpha}))$ is maximum.

(ii) If it is known that $\|\boldsymbol{\alpha}\| = 1$ and $\alpha_1 > 0$, is this maximizer of $V(\boldsymbol{\alpha})$ unique? Justify your answer.

(iii) Show that $V(\boldsymbol{\alpha})$ cannot exceed $n/(n-1)$.

(b) Let $X_1, X_2, \ldots, X_n$ be independent random variables, where $X_i \sim N(\mu_i, \sigma^2)$ for all $i = 1, 2, \ldots, n$. Show that for any $C > 0$, $P(\sum_{i=1}^{n}(X_i - \mu_i)^2 \leq C\sigma^2) \to 0$ as $n \to \infty$.

S8. Let $X_1, X_2, \ldots, X_n$ be independent random variables. Let $E(X_j) = j\theta$ and $V(X_j) = j^3\sigma^2$, $j = 1, 2, \ldots, n$, $-\infty < \theta < \infty$ and $\sigma^2 > 0$. Here $E(X)$ denotes the expectation and $V(X)$ denotes the variance of the random variable X. It is assumed that θ and σ^2 are unknown.

(a) Find the best linear unbiased estimate for θ.

(b) Find the uniformly minimum variance unbiased estimate for θ under the assumption that X_i's are normally distributed; $1 \leq i \leq n$.

S9. Let (X, Y) follow a bivariate normal distribution. Let *mean* of X = *mean* of $Y = 0$, variance of X = variance of $Y = 1$, and the correlation coefficient between X and Y be ρ. Find the correlation coefficient between X^3 and Y^3.

S10. Consider a linear model $Y_i = \mathbf{x}_i^T\boldsymbol{\beta} + \epsilon_i$, $i = 1, 2, \ldots, n$, where $\mathbf{x}_i = (1, x_{i1}, x_{i2}, \ldots, x_{ip})^T$, $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \ldots, \beta_p)^T$, and the superscript T stands for transpose. Assume that the $\mathbf{x}_i$'s are non-stochastic and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent, where $\epsilon_i \sim N(0, \sigma^2 i)$ for all i.

(a) Consider p non-zero real numbers $l_1, l_2, \ldots, l_p$. Derive the condition under which $l_1\beta_1 + l_2\beta_2 + \ldots + l_p\beta_p$ is estimable.

(b) Assuming β_1 is estimable, find the best linear unbiased estimator for β_1.

(c) Is the estimator in (b) the uniformly minimum variance unbiased estimator? Justify your answer.

(d) Check whether the estimator in (b) is consistent.

S11. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed observations with a common exponential distribution with mean μ. Show that there is no uniformly most powerful test for testing $H_0 : \mu = 1$ against $H_A : \mu \neq 1$ at a given level $0 < \alpha < 1$, but there exists a uniformly most powerful unbiased test. Derive that test.

S12. (a) An unbiased die is rolled once. Let the score be $N \in \{1, 2, \ldots, 6\}$. The die is then rolled n times. Let X be the maximum of these n scores. Find the probability of the event $(X = 6)$.

(b) The unit interval (0,1) is divided into two sub-intervals by picking a point at random from the interval. Let Y and Z be the lengths of the longer and shorter sub-intervals, respectively. Find the distribution of Z and show that $\frac{Y}{Z}$ does not have a finite expectation.

S13. Suppose that $X_1, X_2, \ldots, X_{2m}$ are i.i.d. $N(0,1)$. For $k = 1, 2, \ldots, m$, define

$$Y_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{2m} X_j \cos\left(\frac{\pi j k}{m}\right) \text{ and } Z_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{2m} X_j \sin\left(\frac{\pi j k}{m}\right).$$

(a) Show that $(Y_1, Y_2, \ldots, Y_m, Z_1, Z_2, \ldots, Z_m)$ follows a multivariate normal distribution.

(b) Calculate the dispersion matrix of this distribution.

S14. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ are two independent multivariate normal random vectors with $E(\mathbf{X}) = E(\mathbf{Y}) = \mathbf{0}$, $D(\mathbf{X}) = A = ((a_{ij}))$ and $D(\mathbf{Y}) = B = ((b_{ij}))$, where $D(.)$ denotes the dispersion matrix.

(a) Using $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, construct another set of random variables $\mathbf{Z} = (Z_1, \ldots, Z_n)$ such that $E(\mathbf{Z}) = \mathbf{0}$ and $Cov(Z_i, Z_j) = c_{ij} = a_{ij}b_{ij}, 1 \le i, j \le n$, so that $D(\mathbf{Z}) = C = ((c_{ij}))$.

(b) Using (a) or otherwise show that if a_{ii}=b_{ii}=1 for $1 \le i \le n$, then $max(\underline{\lambda}(A), \underline{\lambda}(B)) \le \underline{\lambda}(C) \le \overline{\lambda}(C) \le min(\overline{\lambda}(A), \overline{\lambda}(B))$, where $\underline{\lambda}$ and $\overline{\lambda}$ denote respectively the smallest and largest eigen values of a matrix.

S15. (a) Consider a regression problem, where data $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ are available only at two distinct values of X. Show that if we use the least square method, the models $Y = \beta_0 + \beta_1 X + \epsilon$ and $Y = \alpha_1 X + \alpha_2 X^2 + \epsilon$ will fit the data equally well.

(b) Consider the following regression model

$$y_i = \beta_0 + \beta_1\, x_{1i} + \beta_2\, x_{2i} + \beta_{11}\, x_{1i}^2 + \beta_{22}\, x_{2i}^2 + \beta_{12}\, x_{1i}\, x_{2i} + \epsilon_i,$$

where the predictor variables take the following values

i	1	2	3	4	5	6	7
x_{1i}	1	1	-1	-1	0	0	0
x_{2i}	1	-1	1	-1	0	0	0

i. Show that β_0, β_1, $\beta_{11} + \beta_{22}$ and β_{12} are estimable and find the (non-matrix) algebraic forms for the estimates of these parameters.

ii. Find the standard errors of the estimates of $\beta_{11} + \beta_{22}$ and β_{12}.

S16. A population is made up of items of three types: Type 1, Type 2 and Type 3 in proportions π_1, π_2 and π_3 respectively, where $\pi_1+\pi_2+\pi_3 = 1$. An item is drawn at random and replaced. The process is repeated until all types have been observed for the first time. Let N denote the number of selections until all types have been observed for the first time. For example, $N = 4$ for the outcome (Type 3, Type 2, Type 2, Type 1) and $N = 7$ for the outcome (Type 1, Type 3, Type 3, Type 1, Type 3, Type 1, Type 2).

(a) Show that

$$P(N > n) = \sum_{i=1}^{3}(1 - \pi_i)^n - \sum_{i=1}^{3} \pi_i^n \ \text{ for } \ n = 3, 4, \ldots.$$

(b) Compute $E(N)$ when $\pi_1 = 0.5$, $\pi_2 = 0.3$ and $\pi_3 = 0.2$.

S17. (a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed $N(\mu, 1)$ variables. For any fixed $t \in R$, find the uniformly minimum variance unbiased estimators for $\Phi(t-\mu)$ and $\phi(t-\mu)$, where Φ and ϕ respectively denote the cumulative distribution function and the density function of the standard normal variable.

(b) Let A_θ be the square with vertices as (θ,0), (0,θ), (-θ,0) and (0,-θ). Let f_θ be the uniform distribution over A_θ. Let $(x_1, y_1), (x_2, y_2), \ldots,$ (x_n, y_n) be n independent random samples from f_θ. Find the maximum likelihood estimate for θ, assuming θ is a positive real number.

S18. (a) Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_{10}$ be a random sample of 10 observations from a 100 dimensional multivariate normal distribution with dispersion matrix $\sum = ((\sigma_{ij}))_{100\times100}$. Let $\hat{\sigma}_{ij}$ be maximum likelihood estimate of σ_{ij} based on $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_{10}$ Let $\hat{\sum} = ((\hat{\sigma}_{ij}))_{100\times100}$. Show that $rank\ (\hat{\sum}) \leq 10$.

(b) Suppose that the distribution of a random vector $\mathbf{X} = (X_1,\ X_2,\ X_3,\ X_4)'$ is an equal mixture of two independent normal distributions $N(-\boldsymbol{\alpha}, (1-\rho)\mathbf{I}+\rho\boldsymbol{\alpha}\boldsymbol{\alpha}')$ and $N(\boldsymbol{\alpha}, (1+\rho)\mathbf{I}-\rho\boldsymbol{\alpha}\boldsymbol{\alpha}')$, where $\boldsymbol{\alpha} = (1,1,1,1)'$.

i. Compute the dispersion matrix of this distribution.

ii. Define $Y_1 = a_1X_1 + a_2X_2$ and $Y_2 = a_3X_3 + a_4X_4$. Show that the correlation coefficient between Y_1 and Y_2 cannot exceed $2/3$.

S19. Suppose $\boldsymbol{X} \sim \mathrm{N}_p(\boldsymbol{\theta}, \boldsymbol{I}_p)$, where $\boldsymbol{\theta} \in I\!R^p$, and $\boldsymbol{I}_p$ is the identity matrix of order p. For any fixed $x > 0$, define

$$h(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} > x), \quad \boldsymbol{\theta} \in I\!R^p.$$

Assuming $h(\boldsymbol{\theta})$ depends on $\boldsymbol{\theta}$ only through $\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}$, show that $h(\boldsymbol{\theta})$ is a strictly increasing function of $\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}$.

S20. (a) Suppose independent observations are generated from the linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, 2, \ldots, n$, where $x_1, \ldots, x_n$ are known positive numbers, and $\mathrm{E}(\epsilon_i) = 0\; \forall i, \mathrm{Var}(\epsilon_i) = \sigma^2\; \forall i$. However, we fit the model $y_i = \beta_1 x_i + \epsilon_i$. Show that the bias in the least squares estimate of β_1 is at most $\beta_0/\overline{x}$.

(b) Consider the linear model $(\boldsymbol{Y}, \boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{I}_n)$, where $\boldsymbol{X}$ is an $n \times p$ matrix with entries 0, 1 and -1. Assume that $\boldsymbol{X}$ is of full column rank. Show that if $\hat{\beta}_j$ denotes the best linear unbiased estimate (BLUE) of β_j, then

$$\mathrm{Var}(\hat{\beta}_j) \geq \sigma^2/n.$$

S21. (a) Let A and B be two boxes containing m and n balls (m and n are even integers), respectively. Consider the following sequence of trials. In each trial, a box is selected at random with equal probability; one ball is taken from the selected box and it is placed in the other box. The sequence of trials comes to an end when one of the boxes becomes empty. Find the probability that box A becomes empty.

(b) A d-dimensional random vector $\mathbf{X}$ is called spherically symmetric about $\boldsymbol{\theta} \in R^d$ if $(\mathbf{X}-\boldsymbol{\theta})$ and $H(\mathbf{X}-\boldsymbol{\theta})$ have the same distribution for all orthogonal matrices H.

Show that $\mathbf{X}$ is spherically symmetric about $\boldsymbol{\theta}$ if and only if its characteristic function $\psi(\mathbf{t}) = E(e^{i\mathbf{t}^T\mathbf{X}})$ is of the form $\psi(\mathbf{t}) = e^{i\mathbf{t}^T\boldsymbol{\theta}}g(\|\mathbf{t}\|)$ for g being a function defined on $[0, \infty)$. The superscript T stands for transpose.

S22. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli(p) distribution. We want to test $\mathrm{H}_0 : p = 0.4$ against $\mathrm{H}_1 : p = 0.6$.

(a) Find the most powerful test for testing H_0 against H_1.

(b) Use the Central Limit Theorem to determine the sample size needed so that the two probabilities of error of the test obtained in (a) are both approximately equal to α. Find the result in terms of z_α, the $(1-\alpha)$-th quantile of $N(0,1)$.

S23. (a) Let $\{X_n\}_{n=1}^{\infty}$ be independent and identically distributed as $U(0,\theta)$, where $\theta > 0$. Let $X_{(k)}^{(n)}$ denote the k-th order statistic based on $X_1, \ldots, X_n$. Show that for any fixed k, $nX_{(k)}^{(n)}$ converges in distribution to a non-degenerate random variable as $n \to \infty$.

(b) Let X_1 and X_2 be independent and identically distributed as $U(0,1)$. Find the distribution of $Z = \sqrt{X_1 X_2}$.

S24. Let $\mathcal{X} = \{1, 2, 3, \ldots\}$ be the state space of a Markov chain with the transition probability matrix $P = ((p_{i,j}))$ where $p_{i,1} = 1/i^2$ and $p_{i,i+1} = 1 - 1/i^2$ for all $i \geq 1$.

(a) Derive the stationary distribution of the Markov chain.

(b) Is the Markov chain irreducible? Justify your answer.

(c) Does the Markov chain converge to the stationary distribution for all initial states? Justify your answer.

S25. Suppose that we want to test $H_0 : f(x) = \frac{1}{2}e^{-|x|}$ against $H_1 : f(x) = \frac{1}{\pi}(1+x^2)^{-1}$ based on a single observation X, and we reject H_0 if $|X| > c$.

(a) Find the value of c such that the test has 5% level.

(b) Find the power of the test in (a).

(c) Is this a most powerful (MP) test at 5% level? Justify your answer.

(d) Find an MP critical region at 5% level when the alternative hypothesis is $f(x) = e^{-x}$ for $x > 0$.

S26. Consider a linear regression model containing an intercept. If c is the number of times the i-th row of the design matrix is replicated, then show that the i-th leverage value h_{ii} satisfies $n^{-1} \leq h_{ii} \leq c^{-1}$, where n denotes the number of observations.

S27. Let $p_1(x)$ and $p_2(x)$ denote the probability density functions representing two populations, namely, Class-1 and Class-2 respectively. Let P and $(1 - P)$ be the prior probabilities of the Class-1 and Class-2 respectively. Let

$$p_1(x) = \begin{cases} x - 1 & \text{for } x \in [1, 2] \\ 3 - x & \text{for } x \in (2, 3] \\ 0, & \text{otherwise;} \end{cases}$$

$$p_2(x) = \begin{cases} x - 2 & \text{for } x \in [2, 3] \\ 4 - x & \text{for } x \in (3, 4] \\ 0, & \text{otherwise.} \end{cases}$$

Find the Bayes risk of the optimal Bayes rule for this classification problem.

S28. (a) Let $X_1, \ldots, X_5$ be independent and identically distributed as $N(0, 1)$. Calculate the probability $P(X_1 > X_2X_3X_4X_5)$.

(b) Let X be a random variable taking values $1, 2, \ldots, k$, with probabilities $p_1, p_2, \ldots, p_k$, respectively. Prove that

$$\sum_{i=1}^{k} p_i^2 \leq \max\{p_1, p_2, \ldots, p_k\}.$$

(c) Let $(i_1, i_2, \ldots, i_{2n})$ and $(j_1, j_2, \ldots, j_{2n})$ be two random permutations of $(1, 2, \ldots, 2n)$.

i. Let S denote the number of indices k such that $i_k = 2j_k$. For example, if $n = 3$, then $S = 2$ for the permutations $(1, 4, 5, 3, 2, 6)$ and $(3, 2, 6, 4, 1, 5)$. Determine $E(S)$, the expected value of S.

ii. Let Z be a discrete uniform random variable taking values in $\{1, 2, \ldots, 2n\}$. Show that $E(i_Z/j_Z) \geq 1$.

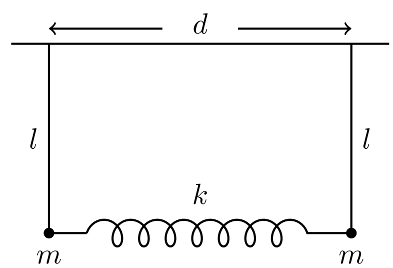
(iii) PHYSICS

P1. A particle of mass m is constrained to move on a smooth sphere of radius a and is acted on by gravity. Choosing θ and ϕ (polar and azimuthal angles, with polar axis vertically up) as the generalized coordinates, write down the Hamiltonian for the particle. Obtain Hamilton's equations of motion and show that

$$a\ddot{\theta} = \frac{k^2 \cos\theta}{m^2 a^3 \sin^3\theta} + g\sin\theta,$$

where k is the constant value of p_ϕ, the generalized momentum corresponding to ϕ.

P2. Two pendulums of mass m and length l are coupled by a massless spring of spring constant k, and are moving in a plane (see figure below). The unstretched length of the spring is equal to the distance d between the supports of the two pendulums.



(a) Set up the Lagrangian in terms of generalized coordinates and velocities.

(b) Derive the equations of motion.

(c) Consider small vibrations and simplify the equations of motion.

(d) Find the frequencies of the two normal modes.

P3. (a) The molar internal energy of a monatomic gas obeying van der Waals equation is given by $\frac{3}{2}RT - \frac{a}{V}$, where a is a constant, and R, T and V carry their usual meanings.

(i) If the gas is allowed to expand adiabatically into vacuum (i.e., free expansion) from volume V_1 to V_2, and T_1 is its initial temperature, what is the final temperature of the gas?

(ii) What would be the final temperature for an ideal gas?

(b) A single classical particle of mass m with energy $\epsilon \leq E$, where $E = P^2/2m$, is enclosed in a volume V; P being the momentum corresponding to the energy E.

(i) Determine the (asymptotic) number of accessible microstates in the energy range ϵ to $\epsilon + d\epsilon$.

(ii) Using this result, obtain the partition function of the aforesaid system if the energy varies between 0 and ∞.

(iii) What would be the free energy of the particle?

P4. A nuclear fission explosion produces a fire ball that can be approximated at some instant to be a black body of 10 cm radius having a temperature $10^8 K$. Calculate

(i) the total electromagnetic power radiated,

(ii) the radiation flux at a distance of 1 km, and

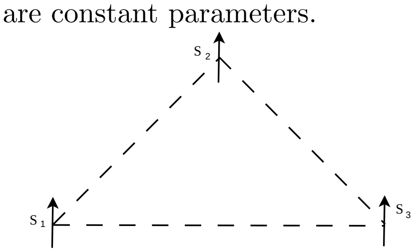
(iii) the wavelength corresponding to maximum energy radiated.

[Stefan's constant (σ)= 0.57×10^{-7} Watt $m^{-2}K^{-4}$ and
Wien's (displacement law) constant = 2.9×10^{-3} meter Kelvin.]

P5. Assume that three spins $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3$ are arranged in the form of an equilateral triangle with each spin interacting with its two neighbors (see figure below). Each spin can take values $+1$ or -1. The energy of this system in a magnetic field perpendicular to the plane of the triangle, is

$$H = -J(\mathrm{S}_1\mathrm{S}_2 + \mathrm{S}_2\mathrm{S}_3 + \mathrm{S}_3\mathrm{S}_1) - \mathrm{F}(\mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3).$$

Here J and F are constant parameters.



(i) Find the partition function of the system.

(ii) Find the average spin.

P6. (a) Consider a simple harmonic oscillator in one dimension with the Hamiltonian $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$, where a and $a^\dagger$ are annihilation and creation operators respectively. The other symbols have their usual meanings.

Show that $a\,|n\!> = \sqrt{n}\,|n-1\!>$, $|n\!>$ being the eigen function of H corresponding to the eigen value $E_n = \hbar\omega(n + \frac{1}{2})$.

(b) Consider a particle of mass m inside a one-dimensional box of length a. Let, at $t = 0$, the state of the particle be given by the following:

$$\psi(x,0) = \begin{cases} A\cos(\frac{\pi x}{a})\sin(\frac{2\pi x}{a}) & 0 \le x \le a \\ 0 & \text{everywhere else.} \end{cases}$$

i. Calculate the value of A.

ii. What is the average energy of the system at $t = 0$.

iii. Write the wave function at $t = T$.

iv. If a measurement for energy is performed at $t = T$, what is the probability of getting the value of the energy equal to $\frac{h^2}{2ma^2}$?

P7. (a) An electron of mass m has the energy of a photon of wavelength λ. Find the velocity of the electron.

(b) Free neutrons have a decay constant k. Consider a non-relativistic situation where the de Broglie wavelength of neutrons in a parallel beam is λ. Determine the distance from the source where the intensity of the beam drops to half of its value at source.

P8. (a) Consider a non-relativistic particle of mass m moving in the potential

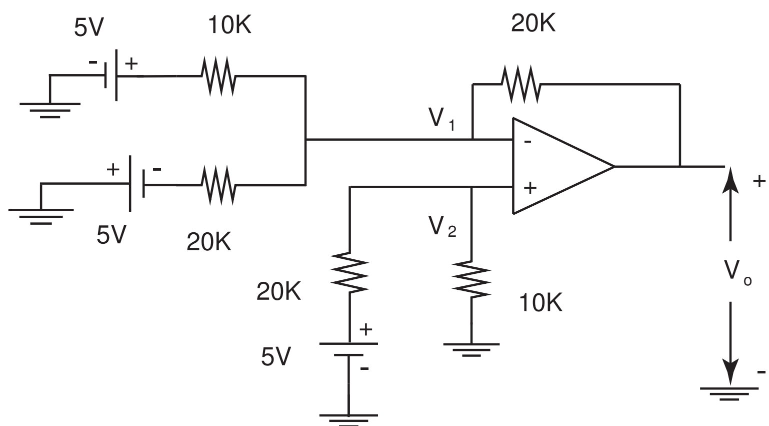
$$V(x) = \begin{cases} V_0 & 0 < x < a \\ \infty & \text{elsewhere} \end{cases}$$

where $V_0 > 0$. Find the energy levels and the corresponding wave functions for all x.

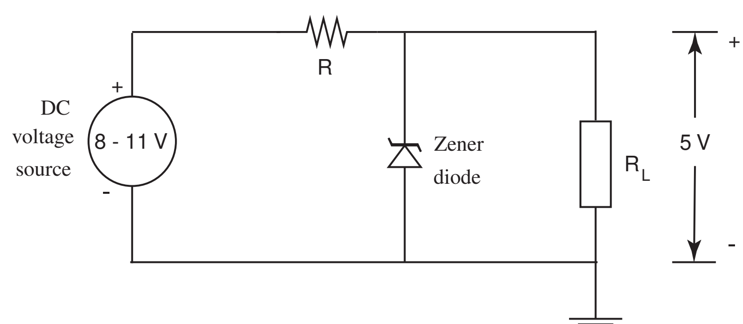
(b) A light beam is propagating through a block of glass with index of refraction μ. If the block of glass is moving at a constant velocity v in the same direction as the beam, what is the velocity of light in the block as measured by an observer in the laboratory?

P9. (a) A negative feedback amplifier has a voltage gain of 100. Variations of the voltage gain up to $\pm 2\%$ can be tolerated for some specific application. If the open-loop gain variations of $\pm 10\%$ are expected owing to spread in device characteristics because of variation in manufacturing conditions, determine the minimum value of the feedback ratio β and also the open loop gain to satisfy the above requirements.

(b) Calculate the output voltage V_0 for the following network:

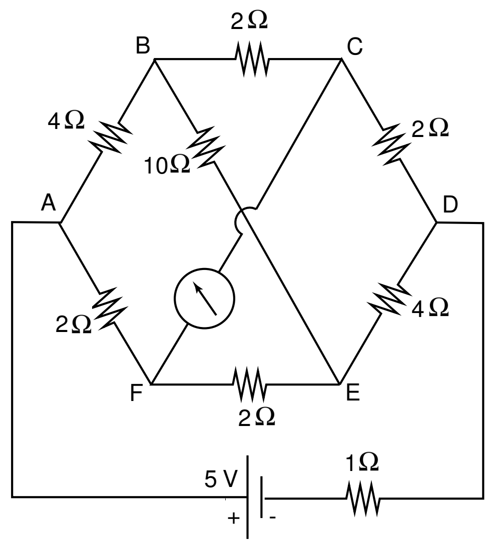


P10. (a) Consider the following circuit for deriving a +5 volt power supply to a resistive load R_L from an input d-c voltage source whose voltage may vary from 8V to 11V. The load R_L may draw a maximum power of 250 mW. The Zener diode has a breakdown voltage of 5 volts. Compute the maximum value of the resistance



R and also the power dissipation requirements for R and the Zener diode. Assume that the minimum breakdown current of the Zener diode is negligible compared to the load current.

(b) Consider the following circuit. Calculate the potential difference between the points F and C, as shown by the ideal voltmeter.



P11. (a) Write the properly normalized Maxwell-Boltzmann distribution $f(u)$ for finding particles of mass m with magnitude of velocity in the interval $[u, u + du]$ at a temperature T.

i. What is the most probable speed at temperature T?

ii. What is the average speed?

iii. What is the average squared speed?

(b) A container is divided into two compartments I and II by a partition having a small hole of diameter d. The two compartments are filled with Helium gas at temperatures $T_1 = 150K$ and $T_2 = 300K$, respectively.

i. How does the diameter d determine the physical process by which the gas comes to steady state?

ii. If λ_1 and λ_2 are the mean free paths in the compartments I and II, respectively, find $\lambda_1 : \lambda_2$ when $d \ll \lambda_1$ and $d \ll \lambda_2$.

iii. Find $\lambda_1 : \lambda_2$ when $d \gg \lambda_1$ and $d \gg \lambda_2$.

P12. (a) i. Give the thermodynamic definition of the Helmholtz free energy F, the classical statistical mechanical definition of the partition function Z, and the relationship between these quantities. Define all the symbols.

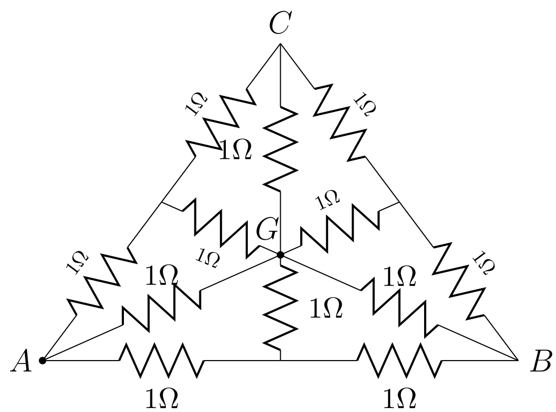
ii. Using these expressions and thermodynamic arguments show that the heat capacity at constant volume C_v is given by

$$C_v = kT\left[\frac{\partial^2}{\partial T^2}(T\ln Z)\right]_v$$

iii. Consider a classical system that has two discrete total energy states E_0 and E_1. Find Z and C_v.

(b) Show that for two dimensional electron gas, the number of electrons per unit area is $n = \dfrac{4\pi m k_B T}{h^2} ln(1+\exp(E_F/k_BT))$, where k_B is the Boltzmann constant, h is the Planck's constant, E_F is the Fermi energy and n is the electron mass.

P13. (a) Determine the equivalent resistance between A and G of the following circuit. Each resistance has the value 1Ω.



(b) Consider a four-input hypothetical logic gate G with the following Karnaugh map.

$AB \backslash CD$	00	01	11	10
00	1			
01		1		
11			1	
10				1

i. Prove or disprove that G can be treated as a universal logic gate. A large number of 0 and 1 lines are available to you.

ii. Implement $A + B$ using minimum number of G gates.

P14. (a) Two very long, thin, cylindrical conducting wires A and B of circular cross-section are kept in air, parallel to each other. Their separation distance (d) is very large compared to their radii (a for both A and B). The wires A and B carry charges $+Q$ and $-Q$ per unit length respectively. Obtain the expression for the capacitance of this system of conductors.

(b) Find the charge and current distributions that would give rise to the potentials

$$\phi = 0, \ \vec{A} = \begin{cases} \dfrac{\mu_0 k}{4c}(ct - |x|)^2 \hat{z} & \text{for } \ |x| < ct \\ \\ 0 & \text{for } \ |x| \geq ct \end{cases}$$

where k is a constant and $c = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$. What would be the surface current?

P15. (a) A substance shows a Raman line at 4567 Å when the exciting line 4358 Å is used. Deduce the positions of Stokes and anti-Stokes lines for the same substance when the exciting line 4047 Å is used.

(b) The ground state of chlorine is ${}^2P_{\frac{3}{2}}$. Find its magnetic moment. How many substates will the ground state split into in a weak magnetic field?

P16. The Λ-particle is an unstable sub-atomic particle that can decay spontaneously into a proton and a negatively charged pion

$$\Lambda \to p + \pi^-.$$

In a certain experiment, the outgoing proton and pion were observed, both travelling in the same direction along the positive x-axis with momenta P_p and P_{π^-} respectively. Find the rest mass of Λ-particle given the rest masses of proton and pion to be m_p and m_{π^-} respectively.

P17. (a) Consider the multiplication of two 2-bit integers $a_1 a_0$ and $b_1 b_0$ to get a 4-bit output $c_3 c_2 c_1 c_0$. Design a circuit for deriving the bit c_2 using only 2-input NAND gates.

(b) Suppose in a voltage sensitive Wheatstone bridge, each arm has a resistance R. Now the resistance of one arm is changed to $R+r$, where $r \ll R$. The Wheatstone bridge is supplied with an input voltage of e_i. Show that on account of imbalance, the output voltage is

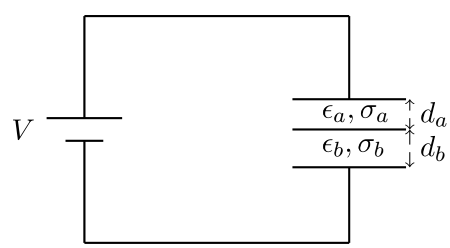
$$\left[\frac{(r/R)}{4+2(r/R)}\right] e_i.$$

P18. (a) Consider a possible solution of Maxwell's equations in vacuum given by

$$\vec{A}(\vec{x},t) = \vec{A}_0 e^{i(\vec{K}.\vec{x}-\omega t)} \quad , \quad \phi(\vec{x},t) = 0.$$

Here $\vec{A}(\vec{x},t)$ is the vector potential and $\phi(\vec{x},t)$ is the scalar potential. $\vec{A}_0$, $\vec{K}$, ω are constants. Show that Maxwell's equations impose the relation, $\mid \vec{K} \mid = \frac{\omega}{c}$, where c is the velocity of the electromagnetic wave in vaccum.

(b) A parallel plate capacitor is filled with two layers of dielectric material a and b (see figure below) and is connected to a battery with potential V. The dielectric constant and conductivity of materials a and b are ϵ_a, σ_a and ϵ_b, σ_b respectively. The thicknesses of the materials a and b are d_a and d_b respectively.



i. Calculate the electric fields in the materials a and b.
ii. Find the current flowing through the capacitor.

P19. A person standing at the rear of a train fires a bullet towards the front of the train. The speed of the bullet, as measured in the frame of the train, is $0.5c$ and the proper length of the train is 400m. The train is moving with speed $0.6c$ as measured by observers on the ground (here c is the velocity of light in vacuum). What do ground observers measure for

(a) the length of the train,

(b) the speed of the bullet, and

(c) the time required for the bullet to reach the front of the train?

P20. (a) Derive the density of states $D(\mathcal{E})$ as a function of energy $\mathcal{E}$ for a free electron gas in one-dimension. (Assume periodic boundary conditions or confine the linear chain to some length L.) Then calculate the Fermi energy $\mathcal{E}_F$ at absolute zero temperature for an N electron system.

(b) The entropy of an ideal paramagnet in a magnetic field is given approximately by

$$S = S_0 - CU^2$$

where U is the energy of the spin system and $C > 0$ is a constant with fixed mechanical parameters of the system.

i. Using the fundamental definition of the temperature (T), determine the energy U of the spin system as a function of T.

ii. Sketch a graph of U versus T for all values of T $(-\infty < T < \infty)$.

P21. (a) Write the wave function for the ground state of hydrogen atom. Calculate the most probable and the average distance of the electron from the nucleus.

(b) Determine the parities of the ground states of nitrogen and oxygen atoms.

P22. (a) A particle of mass m and charge e moves in a magnetic field. The magnetic field is produced by a current I flowing in an infinite straight wire. The wire lies along the z-axis. The vector potential A of the induced magnetic field is given by

$$\begin{aligned} A_r &= A_\theta = 0, \\ A_z &= -(\frac{\mu_0 I}{2\pi}) \ln r, \end{aligned}$$

μ_0 being the permeability of vacuum. r, θ, z are cylindrical coordinates.

(i) Find the Lagrangian of the particle.

(ii) Write down the cyclic coordinates and the corresponding conserved momenta.

(b) In an inertial frame, two events have the space-time coordinates $\{x_1, y, z, t_1\}$ and $\{x_2, y, z, t_2\}$ respectively. Let $(x_2 - x_1) = 3c(t_2 - t_1)$, c representing the velocity of light in vacuum. Consider another inertial frame which moves along x-axis with velocity u with respect to the first one. Find the value of u for which the events are simultaneous in the later frame.

P23. (a) (i) From the first and the second law of thermodynamics prove that for any system

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

The symbols have their usual meanings.

(ii) It is found experimentally that the product of pressure p and the volume V of a given gas is a function of temperature only. Also the internal energy of the gas depends only on temperature. Using (i), obtain the equation of the state of the gas.

(b) Two particles with masses m_1, m_2 and velocities $\vec{v_1}$, $\vec{v_2}$, respectively, stick together after collision. Find the velocity of the composite particle and also the loss in kinetic energy due to the collision.

P24. (a) The nozzle of a bicycle pump is blocked. With no force on the handle, the pump contains a volume V of air at 27^oC and 1 atmospheric pressure. The handle is now pushed down with a constant force F so as to reduce the volume to half. If no air escapes from the pump in the process and the change be adiabatic, compute the final temperature of air in the pump. Assume air to be an ideal gas. γ of air is 1.4 where γ is the ratio of specific heats at constant pressure and volume respectively. Given $2^{1.4} = 2.64$.

(b) Consider a system of N non-interacting quantum oscillators in equilibrium. The energy levels of a single oscillator are $E_n = (n + 1/2)\gamma$, where γ is a constant, $n = 0, 1, 2, \ldots$.

(i) Find the internal energy U and specific heat at constant volume C_v as functions of temperature T.

(ii) Draw a rough sketch of the variation of U and C_v with T.

(iii) Determine the equation of state for the system.

(iv) Calculate the fraction of particles at the n^{th} level.

P25. (a) Write down the Maxwell's equations in SI unit assuming that no dielectric or magnetic material is present. Hence justify what happens to the electric and magnetic fields in the following cases:

(i) If the signs of all the source terms are reversed,

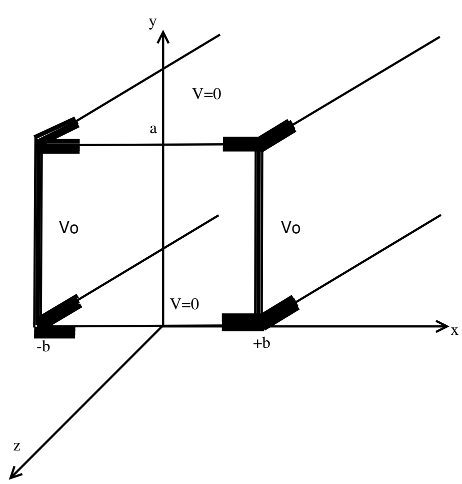
(ii) if the system is space inverted, and

(iii) if the system is time reversed.

(b) (i) A particle moves in a time-independent electric field $\overrightarrow{E} = -\nabla\phi$ and an arbitrary magnetic field $\overrightarrow{B}$, ϕ being the scalar potential. Using Lorentz force law, prove that the energy of the particle is constant.

(ii) If the particle moves along x-axis in the electric field $\overrightarrow{E} = A\exp^{-t/\tau}\,\hat{\imath}$ (where A and τ are constants) and the magnetic field is zero along x-axis, find $x(t)$ with the initial conditions $x(0) = \dot{x}(0) = 0$.

P26. Two infinitely long grounded plates lie parallel to the x-z plane, one at $y = 0$ and the other at $y = a$. They are connected by metal strips maintained at constant potential V_0 as shown in the figure below. A thin layer of insulation at each corner of the metal strips prevents them from shorting out.



(a) Write Laplace's equation to calculate the potential inside the rectangular configuration thus formed.

(b) State the boundary conditions.

(c) Calculate the expression for potential inside the rectangular configuration.

(d) Draw a rough 3D sketch of this expression.

P27. (a) The mean momentum of a particle with wave function $\psi(x)$ is $< p >$. Show that the mean momentum is $< p > +p_o$ for the wave function $e^{ip_0\hat{x}/h}\psi(x)$, where $\hat{x}$ is the position operator and p_0 is a constant.

(b) A particle is in the ground state of a one dimensional box of length L. Suddenly the box expands to twice its size, leaving the wave function undisturbed. Find the probability of finding the particle in the ground state under measurement of Hamiltonian observable.

(c) $\hat{x}$, $\hat{p}$ and $\hat{H}$ are respectively, position, momentum and Hamiltonian operator for harmonic oscillator. Show that $[\hat{x}, \hat{H}] = \frac{i\hbar\hat{p}}{m}$, where m is the mass of the harmonic oscillator.

P28. (a) An electric boiler consists of 2 heating elements, each of 220V, 5 kW rating, connected in series. It contains 10 litres of water at 20^oC. On an average, there is a 20% loss of heat for this boiler. Calculate the time required for the water to boil at normal atmospheric pressure. Calculate the same if the heating elements are connected in parallel. What conclusion can you draw from the ratio of the above two results? [Given J = 4200 Joules/kcal, and the mass of 1 litre of water is 1 kg.]

(b) In a centre-tap full wave rectifier, the load resistance is 5 kΩ. The AC supply across the primary winding is $220 \sin \omega t$. If the transformer turn ratio is 1:2, compute the DC load voltage (V_{DC}) and current (I_{DC}). Assume negligible winding resistance. [Hint: For half wave rectifier, $V_{DC} = V_m \ / \ \pi$.]

(iv) ELECTRICAL AND ELECTRONICS ENGINEERING

E1. (a) Consider the multiplication of two 2-bit integers a_1a_0 and b_1b_0 to get a 4-bit output $c_3c_2c_1c_0$. Design a circuit for deriving the bit c_2 using only 2-input NAND gates.

(b) Consider a bit sequence $a_i,\ i \geq 0$ which has the property $a_{i+4} = a_{i+1} \oplus a_i$.

i. To generate this sequence, which bits of the sequence need to be initialized?

ii. Design a logic circuit using flip-flops and NAND gates to generate the above bit sequence.

iii. For any given value of i, identify the points in the circuit at which the values of $a_i, a_{i+1}, \ldots, a_{i+4}$ may be obtained.

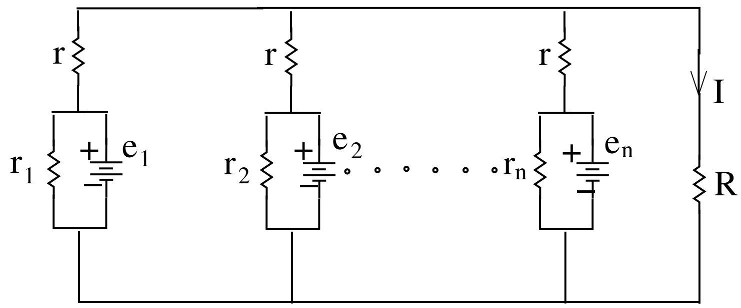
E2. (a) Let us consider a house with two lights; one is at porch gate (porch light denoted by PL) and another is inside the room (room light denoted by RL). The lights are controlled by three switches A, B and C, out of which two are inside the house and one is outside the house. Both the lights are OFF when all the switches are OFF. Both the lights are ON when all the three switches are ON. If any two switches are ON then the porch light is ON. If only one of A, B and C is ON then the light inside the room is ON. Write boolean functions for PL and RL in terms of switch variables A, B and C.

(b) Design a special purpose synchronous counter with not more than 3 flip-flops to provide the following output:

000, 000, 010, 000, 100, 000, 110, 000

You may use additional combinational circuits as needed.

E3. A resistor $\mathbf{R}$ is getting supply from n e.m.f. sources $e_1, e_2, \ldots, e_n$ where $e_1 < e_2 < \cdots < e_n$, connected with corresponding resistors as shown in the figure below.



Calculate the current **I** flowing through the resistor **R**.

E4. (a) A 44 KW, 220 V d.c. machine has 110 Ω shunt resistance and 0.05 Ω armature resistance. Calculate total armature power developed when the machine is working as (i) a generator and (ii) a motor.

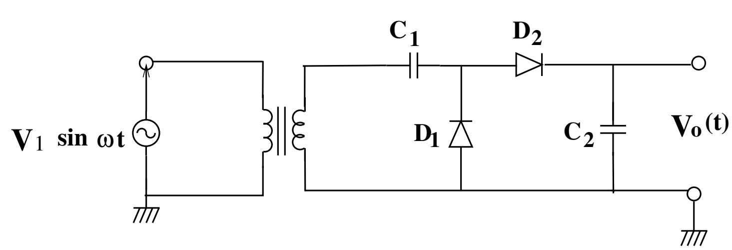
(b) Test data of a 200/400 V, 10 KVA, 50 Hz single phase transformer is as follows.

Short circuit test on secondary side: 20 V, 10 A, 80 W
Open circuit test on primary side: (i) 200 V, 50 Hz, 2500 W
(ii) 100 V, 25 Hz, 1000 W

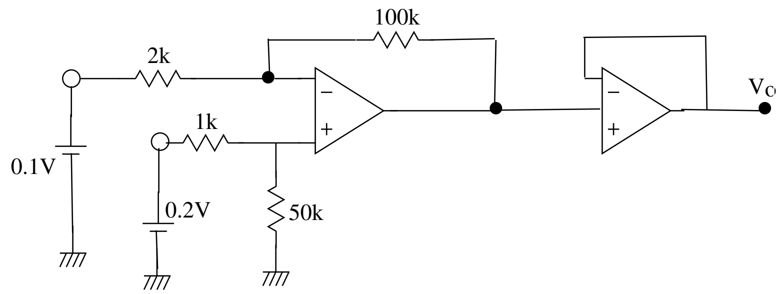
At full-load and unity power factor, calculate:

i. copper loss,
ii. hysteresis loss, and
iii. eddy current loss.

E5. (a) In the following circuit, the diodes D_1 and D_2 and the capacitors C_1 and C_2 are assumed to be ideal. At the input, a sinusoidal voltage $V_1 \sin(\omega t)$ is applied. Sketch the output waveform $V_o(t)$ as a function of time t.



(b) Consider the following circuit with two ideal OP-Amps. Calculate the output voltage, V_o.



E6. Open-circuit and short-circuit tests are conducted on a 220/440 V, 4.4 KVA, single phase transformer. The following readings are obtained.

Open Circuit test with voltage applied on low-voltage side:

Voltage = 110 V,
Current = 1.1A, and
Power = 150 W.

Short Circuit test with voltage applied on high-voltage side:

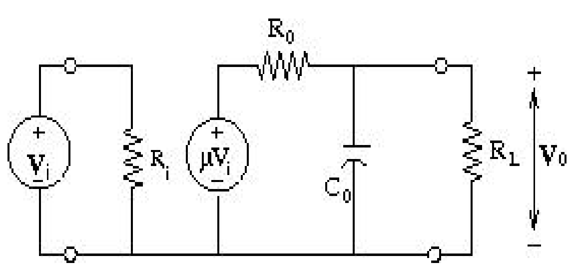
Voltage = 24 V,
Current = 8A, and
Power = 64 W.

At 0.8 p.f. lagging, calculate

(i) the efficiency of the transformer at full-load, and

(ii) the output voltage at the secondary terminal when supplying full-load secondary current.

E7. Consider a voltage amplifier circuit shown in the figure below, where R_i and R_0 represent the input and output impedances respectively, C_0 is the total parasitic capacitance across the output port, μ is the amplifier gain and the output is terminated by a load resistance R_L.



(a) Calculate the current, voltage and power gain in decibels (dB) of the amplifier, when
$R_i = 1M\Omega$, $R_L = 600\Omega$, $R_0 = 100M\Omega$, $C_0 = 10pf$, $\mu = 10$.

(b) Calculate the 3-dB cutoff frequency of the amplifier when
$R_i = 5K\Omega$, $R_L = 1K\Omega$, $R_0 = 100\Omega$, $C_0 = 10pf$, $\mu = 2$.

E8. Assume that an analog voice signal which occupies a band from 300 Hz to 3400 Hz, is to be transmitted over a Pulse Code Modulation (PCM) system. The signal is sampled at a rate of 8000 samples/sec. Each sample value is represented by 7 information bits plus 1 parity bit. Finally, the digital signal is passed through a raised cosine roll-off filter with the roll-off factor of 0.25. Determine

(a) whether the analog signal can be exactly recovered from the digital signal;

(b) the bit duration and the bit rate of the PCM signal before filtering;

(c) the bandwidth of the digital signal before and after filtering;

(d) the signal to noise ratio at the receiver end (assume that the probability of bit error in the recovered PCM signal is zero).

E9. (a) Given a library of 2-input AND, NOT and 2-input XOR gates, synthesize the function $f(A, B, C, D)$ as shown in the Karnaugh map below, using minimum number of gates of the library.

$AB \backslash CD$	00	01	11	10
00	1		1	
01		1	1	
11		1		1
10		1	1	

(b) A sequential lock circuit has two push-buttons **A** and **B** which cannot be pressed simultaneously. It has one output **z** which becomes 1 and opens the lock, only when the buttons are pressed in the sequence **ABBA**. Find a reduced state table for the lock circuit.

E10. Consider the following waveform of a signal $f(t) = 3\cos 500t + 4\cos 1500t$ volts, which is coded using Delta Modulation (DM).

(a) Compute the minimum integer sampling rate with justification for exact reconstruction of the signal from the sampled data.

(b) Assuming a quantizer step size of $\frac{1}{2}$ volt, determine the mean-square quantization noise power.

(c) Determine the cut-off frequency of the low-pass filter in the DM receiver and calculate the corresponding signal-to-noise ratio.

E11. (a) Consider the discrete-time sequence

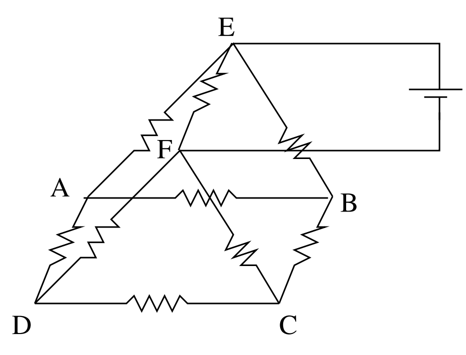
$$x[n] = \begin{cases} (-0.5)^n & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

i. Determine the inverse Z-transform of $X(z^3)$ without computing $X(z)$.

ii. Let $y[n] = e^{j(\pi/3)n}x[n]$. Sketch the pole-zero plot and indicate the Region of Convergence (ROC) of $Y(z)$.

(b) Consider the complex sequence $v[n] = Re\{v[n]\} + jIm\{v[n]\}$. Compute the Z-transform of $Im\{v[n]\}$ in terms of $V(z)$ and indicate the ROC of $Im\{v[n]\}$ in terms of the ROC of $V(z)$.

E12. A prism is made of wire mesh with each side having equal resistance R (see the figure given below). A battery of 6 V and zero internal resistance is connected across E and F. If R is 0.5 Ω, find the current in the branch AD.



E13. (a) Consider a discrete memoryless source with source alphabet $S = \{s_0, s_1, s_2, s_3\}$ with respective probabilities

$$p_0 = \frac{1}{8}; \quad p_1 = \frac{1}{8}; \quad p_2 = \frac{1}{2}; \quad p_3 = \frac{1}{4}.$$

Calculate the probabilities and entropy of the second-order extension of the source.

(b) Suppose a long sequence of information is composed of five possible symbols with probabilities given in the table below:

Symbol	s_0	s_1	s_2	s_3	s_4
Probability	0.4	0.2	0.1	0.1	0.2

Encode the above set of symbols with strings of 0's and 1's based on Huffman coding. Calculate the average code-word length.

E14. (a) A sequence $x[n]$ of length 8 has a Discrete Fourier Transform (DFT) $X[k]$ as follows:

$$X[k] = \begin{cases} 1; & k = 0, 1, 2, 3 \\ k - 2; & k = 4, 5, 6, 7 \\ 0; & \text{otherwise.} \end{cases}$$

$x[n]$ is now up-sampled by a factor of 2 by inserting a zero between every pair of samples of $x[n]$, and appending a zero at the end to produce a sequence $y[n]$ of length 16. Sketch $Y[k]$, the 16-point DFT of $y[n]$.

(b) A zero-mean, wide-sense stationary sequence with variance σ_x^2 passes through a stable and causal all-pass filter with z-transform

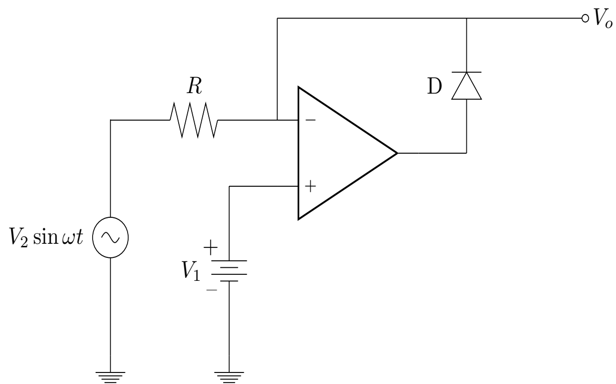
$$H(z) = \frac{\alpha - z^{-1}}{1 - \alpha z^{-1}}.$$

Determine the range of values that α can take. Also calculate the power spectrum of the output, i.e., the Discrete Time Fourier Transform (DTFT) of the auto-correlation sequence of the output.

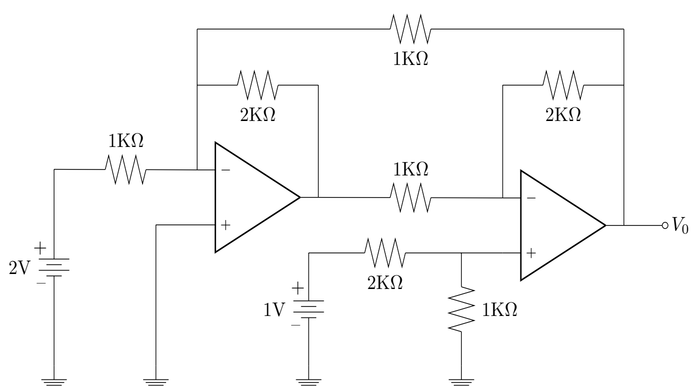
E15. (a) Design a digital circuit to compare two three bit numbers $A = a_2a_1a_0$ and $B = b_2b_1b_0$; the circuit should have three outputs indicating $A = B$, $A < B$ and $A > B$. Give a gate-level diagram of this comparator circuit.

(b) Using two such comparator circuits and as few basic gates as possible, design a digital circuit to compare two six bit numbers $A = a_5a_4a_3a_2a_1a_0$ and $B = b_5b_4b_3b_2b_1b_0$. This circuit should also have only three outputs indicating $A = B$, $A < B$ and $A > B$.

E16. (a) Consider an OP-AMP circuit with a diode D, resistance R and a d.c. source voltage V_1 as shown below. At the input, a voltage signal $V_2 \sin \omega t$ is applied, where $V_2 > V_1$. Trace the plot of output voltage V_0 against time. Assume that both the OP-AMP and the diode are ideal.



(b) Consider the following circuit with two ideal OP-AMPs. The values of the resistances and input sources are shown in the figure. Calculate the output voltage V_0. Show your analysis and justify your argument.

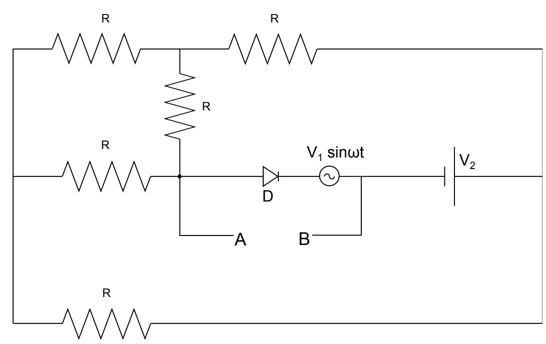


E17. A 22 KVA, 2200/220V two-winding transformer is converted to an auto-transformer with additive polarity.

(a) Calculate the percent increase in KVA of the auto-transformer with respect to the original two-winding transformer.

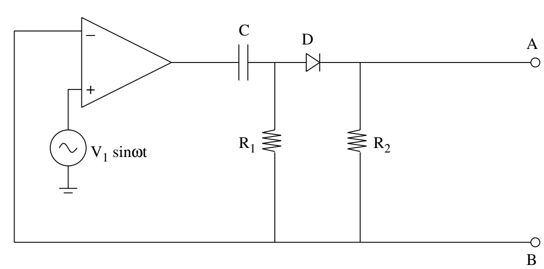
(b) The auto-transformer has a full-load efficiency of 90% at unity power factor. Calculate the efficiency of the auto-transformer when the load is reduced to half at the same power factor. Assume that iron loss is 100 W.

E18. Consider the following circuit, where a sinusoidal source $V_1 \sin \omega t$ and a DC source V_2 are connected as shown. Assume $V_1 > V_2$. Let V_{AB} be the voltage between A and B. The value of each resistance is R.



The voltage sources and the diode D are assumed as ideal. Draw the waveform of V_{AB} and justify your answer.

E19. Consider the following circuit. An a.c. source $V_1 \sin \omega t$ is connected to the non-inverting input of the OP-AMP. Draw the voltage waveform V_{AB} and justify your answer.



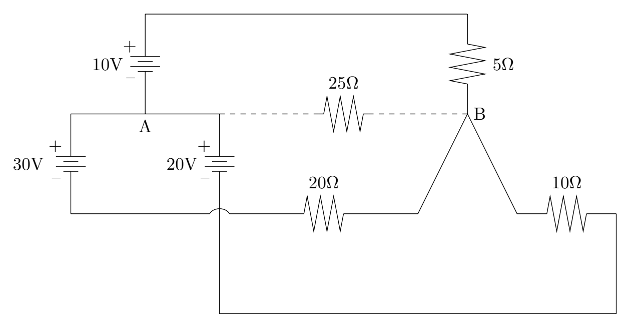
E20. Two d.c. generators A and B are connected to a common load. A has constant $e.m.f.$ of $400V$ and internal resistance 0.25Ω, while B has constant $e.m.f.$ of $410V$ and internal resistance 0.40Ω.

(a) Calculate the current and power output from each generator to the common load having terminal voltage of $390V$.

(b) What change in operation will happen when the common load is open circuited?

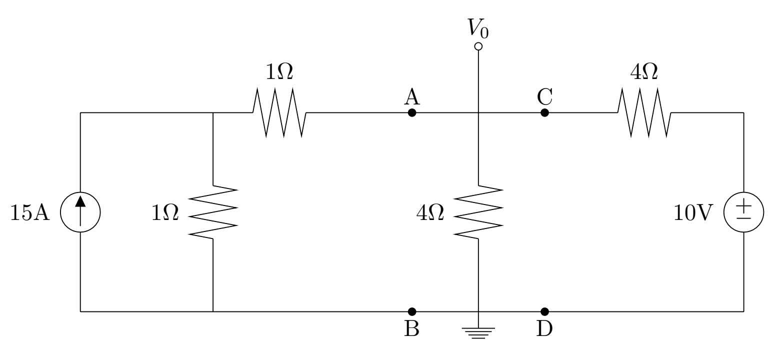
(c) Under the open circuit condition, calculate the current, terminal voltage and energy output from each generator.

E21. (a) Use Norton's theorem in the circuit given below to find the voltage between the points A and B.



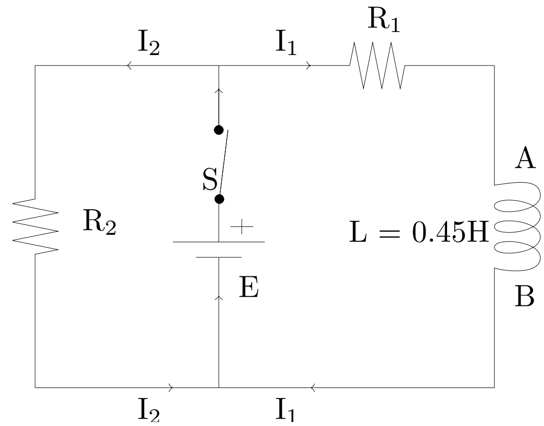
(b) If a 25Ω resistance is connected between A and B, what will be the current through the resistor?

(c) Using the Norton's theorem, find the voltage V_0 in the following figure.



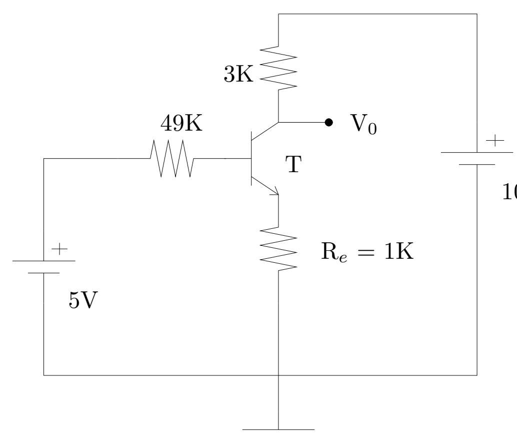
E22. Consider the following circuit. The emf E = 15V, $R_1 = 1.5K\Omega$, and $R_2 = 3.0K\Omega$. The switch is closed for $t < 0$, and steady-state conditions are established. The switch is now thrown open at $t = 0$.

(a) Find the initial voltage emf_0 across L just after $t = 0$. Which end of the coil is at higher potential: A or B?



(b) Draw freehand graphs of the currents in R_1 and R_2 as a function of time, indicating the values before and after $t = 0$.

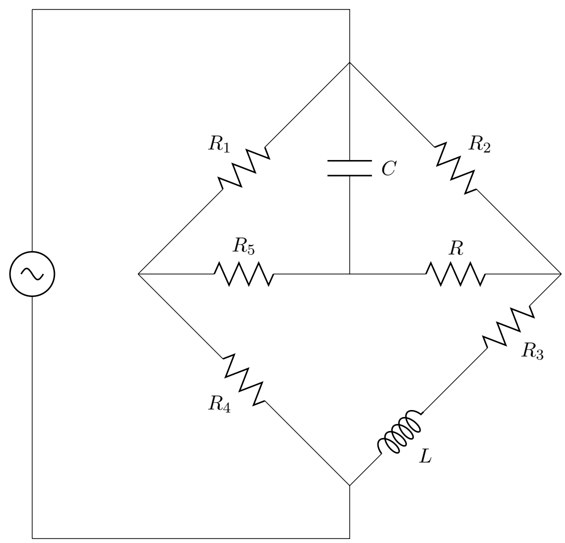
E23. For the circuit shown below, assume $\beta = h_{FE} = 100$, $V_{CE,\,sat} = 0.2V$, $V_{BE,\,sat} = 0.8V$, $V_{BE,\,active} = 0.7V$, and $V_{BE,\,cutoff} = 0.0$. [All symbols follow the standard notations.]



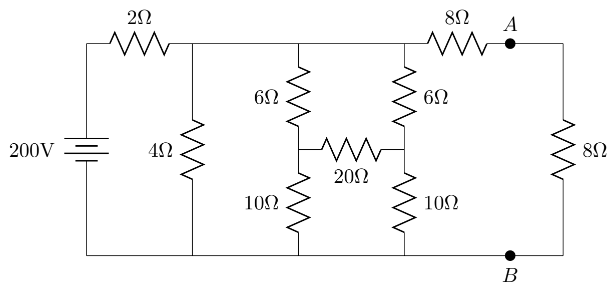
(a) Determine whether the transistor T is in the *cutoff*, *saturation* or *active* region.

(b) Find the minimum value of R_e for which the transistor is in the active region. [Assume $I_{CO} \ll I_B$]

E24. (a) Given the circuit shown below, find the condition under which the current through R will be zero.



(b) Find the current in 8Ω load across AB of the following circuit.

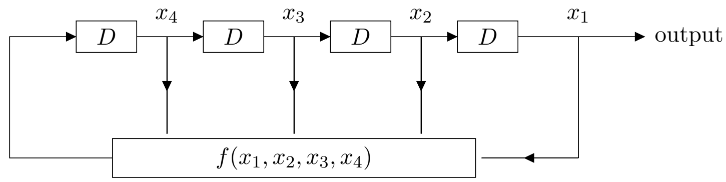


E25. (a) Consider two 2-bit unsigned integers $A = a_1 a_0$ and $B = b_1 b_0$. We like to compute $(A \cdot B) \bmod 3$.

(i) How many bits are required to represent the result?

(ii) Design a Boolean circuit using only two-input NAND gates that accepts a_0, a_1, b_0 and b_1 as inputs and computes the result.

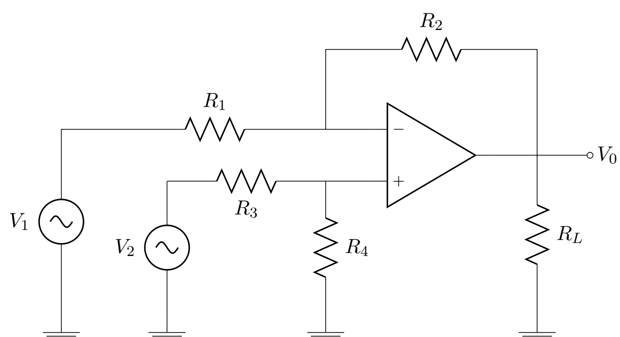
(b) Consider the synchronous circuit consisting of four D flip-flops as shown in the figure below. All the flip-flops are driven by the same clock signal (not explicitly shown in the figure). This circuit generates a periodic binary output sequence '1100101000' repeatedly with the leftmost bit appearing first at the output. Initially, $x_1 = 1$, $x_2 = 1$, $x_3 = 0$ and $x_4 = 0$. Find a minimal expression for the combinational circuit $f(x_1, x_2, x_3, x_4)$.



E26. (a) The equivalent noise resistance for a multi-stage amplifier is the input resistance that will produce the same random noise at the output of the amplifier as the actual amplifier does.

The first stage of a two-stage amplifier has a voltage gain of 10, a 600Ω input resistance, a 1600Ω equivalent noise resistance and a 30 kΩ output resistance. For the second stage, these values are 25, 80 kΩ, 10 kΩ and 1 MΩ, respectively. Calculate the equivalent input noise resistance of this two-stage amplifier.
For an input noise resistance R, the noise voltage E_n generated at the input of a circuit is $E_n = \sqrt{4kT\Delta fR}$, where k is the Boltzmann constant, T is the temperature and Δf is the bandwidth.

(b) Consider the circuit shown below.



(i) What would be the value of V_0 for an ideal OP-AMP if $R_1 = R_2 = R_3 = R_4$?

(ii) What type of amplifier is represented by the above circuit?

E27. (a) Consider that only the four alphabets a, b, c, d are used in a communication between two parties. Studying a good amount of data, it has been noted that the expected probabilities of occurrences of these four alphabets are 0.41, 0.31, 0.21 and 0.07, respectively. You are required to encode these alphabets as binary strings so that the average length of the encoded bitstream is minimum. The lengths of the codewords may be different.

The encoded binary string will not have any separating character between any two codewords and the recipient should be able to decode the binary string to revert back to the alphabets sent.

(i) Write down the four binary codewords corresponding to a, b, c and d.

(ii) Explain how you arrive at them.

(iii) What is the average bit-length per alphabet for your coding scheme?

(b) A frequency modulated (FM) signal is represented as

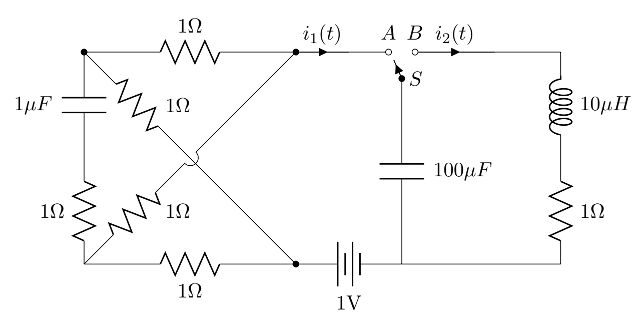
$$e = 12\sin(1.8\pi \times 10^8 t + 5\sin 400\pi t).$$

Find the carrier and modulating frequencies, and the maximum deviation of the FM wave. What power will this FM wave dissipate in a 10Ω resistor?

E28. (a) Consider the following circuit where the capacitors and the inductor are ideal. At time $t = 0$, the switch S is open and at time $t = 1$, S is connected to node A. At time $t = 2$, S is disconnected from node A and connected to node B.

(i) Write an expression of the current $i_1(t)$.

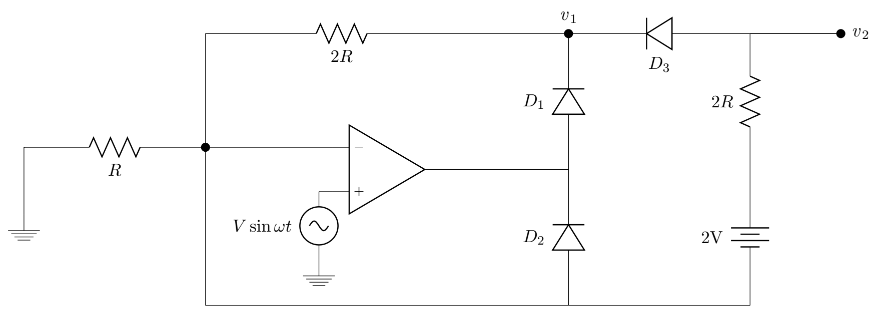
(ii) Plot the waveform of $i_2(t)$ against time t.



(b) Consider the following circuit with three resistors, three diodes D_1, D_2, D_3 and one AC source $V\sin wt$. Assume that the OP-AMP and the diodes are ideal.

(i) Show the voltage waveforms at v_1 and v_2 with respect to time t.

(ii) What is the maximum value of v_1 and v_2?



E29. (a) A 50 Hz, 4-pole, 3-phase induction motor is running at 1200 rpm. It is connected to a 440V power line. It requires power input of 33 kW at 0.8 power factor lag. The motor has stator loss 1 kW and windage and friction loss 1.6 kW.

(i) Find the rotor copper loss.

(ii) Find the efficiency of the induction motor.

(b) A shunt generator delivers 100 kW at 250V and 400 rpm. The armature resistance is 0.02Ω and field resistance is 50Ω. The total contact drop for the brush is 2V. Calculate the speed of the machine running as a shunt motor and taking 50 kW input at 250V.

E30. (a) A single phase 440/220V transformer has an effective primary resistance of 1Ω and secondary resistance of 0.11Ω. Its iron loss on normal input potential is 64W. Calculate maximum efficiency of the transformer at unity power factor.

(b) Let $f(x),\ x = 0, 1, 2, \ldots, N$-1, be real periodic input data with periodicity N. Let $F(u), u = 0, 1, 2, \ldots, N$-1, be the discrete Fourier transform of $f(x)$. Compute the Fourier coefficients and phase angles for $f(x) = [10, 5, 7, 11, 17, 11, 7, 5]$ and $N = 8$, and indicate if these coefficients reflect any special property.

(v) COMPUTER SCIENCE

C1. (a) Write the smallest real number greater than 6.25 that can be represented in the IEEE-754 single precision format (32-bit word with 1 sign bit and 8-bit exponent).

(b) Convert the sign-magnitude number **10011011** into a 16-bit 2's complement binary number.

(c) The CPU of a machine is requesting the following sequence of memory accesses given as word addresses: 1, 4, 8, 5, 20, 17, 19, 56. Assuming a direct-mapped cache with 8 one-word blocks, that is initially empty, trace the content of the cache for the above sequence.

C2. (a) A machine $\mathcal{M}$ has the following *five* pipeline stages; their respective time requirements in nanoseconds (ns) are given within parentheses:
F-stage — instruction fetch (9 ns),
D-stage — instruction decode and register fetch (3 ns),
X-stage — execute/address calculation (7 ns),
M-stage — memory access (9 ns),
W-stage — write back to a register (2 ns).
Assume that for each stage, the pipeline overhead is 1 ns. A program P having 100 machine instructions runs on $\mathcal{M}$, where every 3^{rd} instruction needs a 1-cycle stall before the X-stage. Calculate the CPU time in seconds for completing P.

(b) The CPU of a computer has a ripple-carry implementation of a 2's complement adder that takes two 8-bit integers $A = a_7a_6 \ldots a_0$ and $B = b_7b_6 \ldots b_0$ as inputs, and produces a sum $S = s_7s_6 \ldots s_0$, where $a_i,\ b_i,\ c_i \in \{0, 1\}$ for $(0 \leq i \leq 7)$.
Let $A = 1001\ 1001$ and $B = 1000\ 0110$. What will be the output S of the adder? How will the value of S be interpreted by the machine?

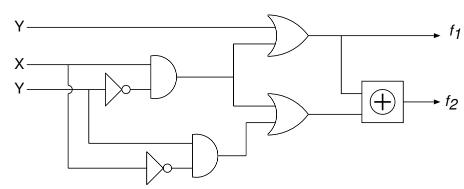
(c) Add the following two floating point numbers A and B given in IEEE 754 single precision format and show the sum S in the same format.
A: 0000011000100 0000 000000000000001
B: 1000011000100 0000 000000000000001

C3. Draw a complete binary tree T with $(N-1)$ nodes where $N = 2^n$. Suppose each node in T is a processor and each edge of T is a physical link between two processors through which they can communicate. Given M arrays $A_i = \{e_{1i}, e_{2i}, \ldots, e_{Ni}\}$ for $1 \le i \le M$, develop an algorithm for the given architecture to compute the sum of each array $SUM_i = \sum_{j=1}^{N} e_{ji}$ for all i in $O(\log N + M)$ time.

C4. Let C denote a logic block that is capable of comparing two 4-bit 2's complement numbers A (a_3, a_2, a_1, a_0) and B (b_3, b_2, b_1, b_0), where $a_i, b_i \in \{0, 1\}$ for $i = 0, 1, 2, 3$. The circuit C has eight input lines $a_3, a_2, a_1, a_0, b_3, b_2, b_1, b_0$, and three output lines E, L, G (Equal: E; Less than: L; Greater than: G). For example, if $A > B$, then the outputs should be $E = 0, L = 0$, and $G = 1$.

Write the Boolean equations for the three outputs E, L, and G.

C5. (a) A Boolean function f is said to be positive unate if f can be expressed in a form where all the variables appear in uncomplimented form, and only the AND and OR Boolean operators are used. For example, the function $g_1 = X_1X_2 + X_2X_3$ is positive unate but $g_2 = X_1X_2 + \bar{X}_2X_3$ is not. Consider the following circuit and determine which of the two functions f_1 and f_2 are positive unate.



(b) Consider a machine with four registers (one of which is the accumulator `A`) and the following instruction set.

- `LOAD R` and `STORE R` are indirect memory operations that load and store using the address stored in the register `R`. Thus, `LOAD R` loads the contents of memory[R] into `A` and `STORE R` stores the contents of `A` in memory[R].
- `MOV R1 R2` copies the contents of register `R1` into register `R2`.
- `ADD R` and `SUB R` operate on the accumulator and one other register `R`, such that `A = A op R`.
- `LDC n` stores the 7-bit constant `n` in the accumulator.

- `BRA`, `BZ`, and `BNE` are branch instructions, each taking a 5-bit offset.

Design an instruction encoding scheme that allows each of the above instructions (along with operands) to be encoded in 8 bits.

C6. (a) In a Buddy memory allocation system, a process is allocated an amount of memory whose size is the smallest power of 2 that is greater than or equal to the amount requested by the process.

A system using buddy memory allocation has 1MB memory. For a given sequence of nine processes, their respective memory requirements in KB are:
50, 150, 90, 130, 70, 80, 120, 180, 68.

(i) Illustrate with an allocation diagram to justify whether all the requests, in the given order, can be complied with. Assume that memory once allocated to a process is no longer available during the entire span of the above sequence.

(ii) Calculate the total memory wasted due to fragmentation in your memory allocation by the above scheme.

(b) Two processes P_1 and P_2 have a common shared variable *count*. While P_1 increments it, P_2 decrements it. Given that R_0 is a register, the corresponding assembly language codes are:

P_1:	$count++$		P_2:	$count--$	
MOV	*count*	R_0	MOV	*count*	R_0
ADD	#1	R_0	SUB	#1	R_0
MOV	R_0	*count*	MOV	R_0	*count*

Give an example to justify whether a race condition may occur if P_1 and P_2 are executed simultaneously.

C7. (a) Five batch jobs $P_1, \ldots, P_5$ arrive almost at the same time. They have estimated run times of 10, 6, 2, 4 and 8 ms, respectively. Their priorities are 3, 5, 2, 1 and 4 respectively, where 1 indicates the highest priority and 5 indicates the lowest. Determine the average turnaround and waiting time for the following scheduling algorithms:

(i) Round robin with time quantum of 5 ms,

(ii) Priority scheduling.

(b) The access time of a cache memory is 100 ns and that of main memory is 1000 ns. It is estimated that 80% of the memory requests are for read and the remaining 20% are for write. The hit ratio for read access is 0:9. A write through procedure is used.

(i) What is the average access time of the system considering only memory read cycles?

(ii) What is the average access time of the system considering both read and write requests?

C8. (a) What are the conditions which must be satisfied by a solution to the critical section problem?

(b) Consider the following solution to the critical section problem for two processes. The two processes, P_0 and P_1, share the following variables:

```
char flag[2] = {0,0};
char turn = 0;
```

The program below is for process P_i ($i = 0$ or 1) with process P_j ($j = 1$ or 0) being the other one.

```
do {
 flag[i] = 1;
 while (flag[j])
       if (turn == j) {
          flag[i] = 0;
          while (turn == j) {};
       }
 ...
 CRITICAL SECTION
 ...
 turn = j;
 flag[i] = 0;
 ...
 REMAINDER SECTION
 ...
} while (1);
```

Does this solution satisfy the required conditions?

C9. (a) Suppose that an operating system provides two functions, `block()` which puts the calling process on the blocked queue, and `wakeup(P)` which moves process P to the runnable queue if it is currently on the blocked queue (otherwise, its behaviour is unpredictable).

Consider two processes A and B running the code given below.

The intended behaviour of the code is to have A and B run forever, alternately printing their names on the screen.

```
void A()                        void B()
{ while(1) {                    { while(1) {
    block();                        printf("B");
    printf("A");                    wakeup(A);
    wakeup(B);                      block();
  }                               }
}                               }
```

i. Construct a scenario in which the intended behaviour would not be observed.

ii. Redesign the code using semaphore(s) so that it works correctly. You should show the initialisation of the semaphore(s), and the calls to `wait()` and `signal()` made by A and B.

(b) A system has 4 processes A, B, C, D and 5 allocatable resources R_1, R_2, R_3, R_4, R_5. The maximum resource requirement for each process and its current allocation are as follows.

Process	Maximum	Allocation
	R_1, R_2, R_3, R_4, R_5	R_1, R_2, R_3, R_4, R_5
A	1, 1, 2, 1, 3	1, 0, 2, 1, 1
B	2, 2, 2, 1, 0	2, 0, 1, 1, 0
C	2, 1, 3, 1, 0	1, 1, 0, 1, 0
D	1, 1, 2, 2, 1	1, 1, 1, 1, 0

Suppose the currently available count of resources is given by $0, 0, X, 1, 1$. What is the minimum value of X for which this is a safe state? Justify your answer.

C10. (a) The C function `divby3` given below is intended to check whether a given number is divisible by 3. It assumes that the argument `number` is a string containing the decimal representation of a positive integer, and returns 1 or 0 depending on whether the integer is divisible by 3 or not.

```
int divby3(char *number)
{
    int sum = 0;
    while (*number != '\0') {
        sum += *number - '0';
        number++;
```

```
    }
    return (sum % 3) ? 0 : 1;
}
```

Assume that a variable of type `int` is stored using 4 bytes and the decimal representations of arbitrarily large positive integers can be passed as arguments to `divby3`.

i. Show that the given function does not work correctly for some integers larger than 10^{10^9}.

ii. Modify the above function so that it works as intended for all positive integers.
NOTE: The smaller the number of ALU operations used by your function, the more marks you will get.

(b) There are n students standing in a line. The students have to rearrange themselves in ascending order of their roll numbers. This rearrangement must be accomplished only by successive swapping of adjacent students.

i. Design an algorithm for this purpose that minimises the number of swaps required.

ii. Derive an expression for the number of swaps needed by your algorithm in the worst case.

C11. Consider the *fast square and multiply algorithm* to calculate $x^y \bmod N$ as given below, where x, y, N are positive integers and $1 \leq x, y < N$.

Input: x, y, N
Output: $x^y \bmod N$
$z = y, u = 1, v = x$;
while $z > 0$ **do**
    if $z \equiv 1 \; mod \; 2$ **then**
        $u = uv \bmod N$;
    end
    $v = v^2 \bmod N$; $z = \lfloor \frac{z}{2} \rfloor$;
end
return u.

(a) Write a C function to implement the algorithm. Your function should take three arguments x, y and N, and return the value $x^y \bmod N$, all of which are of type `unsigned long long` (i.e., 64-bit unsigned integers).

(b) Discuss whether your program works perfectly for all possible input combinations.

(c) What is the time complexity of the algorithm (not your C implementation) in terms of N? [Note that N can be a very large integer, e.g., more than 512 bits. Assume that the time complexity of modular multiplication is $O(\log^2 N)$, when the positive integers involved are less than N.]

C12. (a) Let $R = (A, B, C, D, E, F)$ be a schema with the set $F = \{A \to BC,\ CD \to E, B \to D, E \to A\}$ of functional dependencies. Suppose R is decomposed into two schemata $R_1 = (A, B, C)$ and $R_2 = (A, D, E, F)$

(i) Is this decomposition loss-less? Justify.

(ii) Is this decomposition dependency preserving? Justify.

(iii) Identify all the candidate keys for R.

(iv) Decompose R into normalized sets of relations using 3NF.

(v) If a new dependency $A \twoheadrightarrow F$ (multi-valued dependency) is introduced, what would be the new set of normalized relations?

(b) Consider the relations $r1(A, B, C)$, $r2(C, D, E)$ and $r3(E, F)$. Assume that the set of all attributes constitutes the primary keys of these relations, rather than the individual ones. Let $V(C, r1)$ be 500, $V(C, r2)$ be 1000, $V(E, r2)$ be 50, and $V(E, r3)$ be 150, where $V(X, r)$ denotes the number of distinct values that appear in relation r for attribute X. If r_1 has 1000 tuples, r_2 has 1500 tuples, and r_3 has 750 tuples, then give the ordering of the natural join $r_1 \bowtie r_2 \bowtie r_3$ for its efficient computation. Justify your answer.

C13. A block of bits with n rows and m columns uses horizontal and vertical parity bits for error detection. If exactly 4 bits are in error during transmission, derive an expression for the probability that the error will be detected.

C14. A school database maintains the following relations for its students, teachers and subjects:

- Student(st_name, st_address, class, section, roll_no, regn_no)
- Teacher(t_name, t_address, tel_no)

- Subject(s_name, t_name, text_book, class)

Consider the following constraints on the existing data.

- A student after admission to the school is assigned with a unique regn_no. However, a student also gets a roll_no that starts from 1 for each class and section. A class can have many sections and a student is placed in only one class and section as expected in a school.
- In the school a teacher's name (t_name) has been found to be unique. However, more than one teacher may stay at the same address and the tel_no is a land line connection where an address will have only one such telephone.
- A subject name (s_name) is unique but the same subject may be taught in many classes (for example, History may be taught in many classes with different contents but s_name remains the same). Every subject has a set of standard text_books for a class and there may be more than one teacher who can teach the subject. Any teacher may use any of the standard text books to teach a subject.

(a) Considering the above constraints, identify the functional /multivalued dependencies present and normalize the relations.

(b) Using the normalized set of relations answer the following query using relational algebra or SQL: List all the teachers (t_name) who can teach History in Class V and reside in "Baranagar" (name of a locality). Consider that any address offers a locality name.

C15. A program P consisting of 1000 instructions is run on a machine at 1 GHz clock frequency. The fraction of floating point (FP) instructions is 25%. The average number of clock-cycles per instruction (CPI) for FP operations is 4.0 and that for all other instructions is 1.0.

(a) Calculate the average CPI for the overall program P.

(b) Compute the execution time needed by P in seconds.

C16. Consider a 100mbps token ring network with 10 stations having a ring latency of 50 μs (the time taken by a token to make one complete rotation around the network when none of the stations is active). A station is allowed to transmit data when it receives the token, and it releases the token immediately after transmission. The maximum allowed holding time for a token (THT) is 200 μs.

(a) Express the maximum efficiency of this network when only a single station is active in the network.

(b) Find an upper bound on the token rotation time when all stations are active.

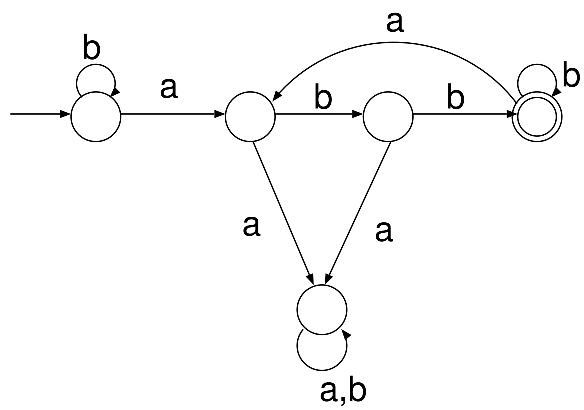
(c) Calculate the maximum throughput rate that one host can achieve in the network.

C17. (a) Station A is sending data to station B over a full duplex error free channel. A sliding window protocol is being used for flow control. The send and receive window sizes are 6 frames each. Each frame is 1200 bytes long and the transmission time for such a frame is 70 μS. Acknowledgment frames sent by B to A are very small and require negligible transmission time. The propagation delay over the link is 300 μS. What is the maximum achievable throughput in this communication?

(b) Consider a large number of hosts connected through a shared communication channel. The hosts transmit whenever they have any data. However, if two data packets try to occupy the channel at the same time, there will be a collision and both will be garbled. The hosts retransmit these packets that suffered collisions. Assume that the generation of new packets by a host is a Poisson process and is independent of other hosts. The total number of transmissions (old and new packets combined) per packet time follows Poisson distribution with mean 2.0 packets per packet time. Compute the throughput of the channel. (Packet time is the amount of time needed to transmit a packet.)

C18. (a) Construct a finite state machine that accepts all the binary strings in which the number of 1's and number of 0's are divisible by 3 and 2, respectively.

(b) Describe the language recognized by the following machine.



(c) Consider the grammar $E \to E + n | E \times n | n$. For a sentence $n + n \times n$, find the handles in the right sentential form of the reductions.

C19. Design a Turing machine that recognizes the unary language consisting of all strings of 0's whose length is a power of 2, i.e., $L = \{0^{2^n} | n \geq 0\}$.

C20. (a) Write a context-free grammar for the language consisting of all strings over $\{a, b\}$ in which the number of a's is not the same as that of b's.

(b) Let the set $D = \{p \mid p$ is a polynomial over the single variable x, and p has a root which is an integer (positive or negative)$\}$.

i. Design a Turing machine (TM) to accept D.

ii. Can D be decided by a TM? Justify your answer.

C21. (a) Give a context-free grammar G that generates $L = \{0^i 1^j 0^k \mid i + k = j\}$. Prove that $L = L(G)$.

(b) Write a regular expression for all strings of 0's and 1's in which the total number of 0's to the right of each 1 is even. Justify your answer.

C22. Let $S = \{x_1, x_2, \ldots x_n\}$ be a set of n integers. A pair (x_i, x_j) (where $i \neq j$) is said to be the closest pair if $|x_i - x_j| \leq |x_{i'} - x_{j'}|$, for all possible pairs $(x_{i'}, x_{j'})$, $i', j' = 1, 2, \ldots, n, i' \neq j'$.

(a) Describe a method for finding the closest pair among the set of integers in S using $O(n \log_2 n)$ comparisons.

(b) Now suggest an appropriate data structure for storing the elements in S such that if a new element is inserted to the set S or an already existing element is deleted from the set S, the current closest pair can be reported in $O(\log_2 n)$ time.

(c) Briefly explain the method of computing the current closest pair, and necessary modification of the data structure after each update. Justify the time complexity.

C23. Let A be an $n \times n$ matrix such that for every 2×2 sub-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of A, if $a < b$ then $c \leq d$. Moreover, for any pair of rows i and j, if a_{ik} and a_{jl} are the largest elements in i-th and j -th rows of A, respectively, then $k \leq l$ (as illustrated in the 5×5 matrix below).

$$\begin{bmatrix} 3 & 4 & 2 & 1 & 1 \\ 7 & 8 & 5 & 6 & 4 \\ 2 & 3 & 6 & 6 & 5 \\ 5 & 6 & 9 & 10 & 7 \\ 4 & 5 & 5 & 6 & 8 \end{bmatrix}$$

(a) Write an algorithm for finding the maximum element in each row of the matrix with time complexity $O(n \log n)$.

(b) Establish its correctness, and justify the time complexity of the proposed algorithm.

C24. Let M be an $(n \times n)$ matrix where each element is a distinct positive integer. Construct another matrix M' by permuting the rows and/or permuting the columns, such that the elements of one row appear in *increasing* order (while looking from left to right) and those of one column appear in *decreasing* order (while looking from top to bottom).

(a) Describe an $O(n^2)$ time algorithm for constructing M'. Justify your analysis.

(b) Propose a data structure that supports your algorithm. Clearly explain how much additional storage, other than the matrix itself, is required in your algorithm.

C25. A connected, simple, undirected planar graph $G(V, E)$ is given where V denotes the set of vertices and E denotes the set of edges. In V, there is a designated source vertex s and a designated destination vertex t. Let $P(v)$ denote the shortest path (may contain repetition of nodes/edges) from s to t that passes through v, and let $l(v)$ denote the path length (i.e., the number of edges) of $P(v)$.

(a) Describe an $O(|V|)$ time algorithm that determines the value of τ where $\tau = \max_{\forall v \in V} l(v)$. Justify your analysis.

(b) Propose a data structure that supports your algorithm.

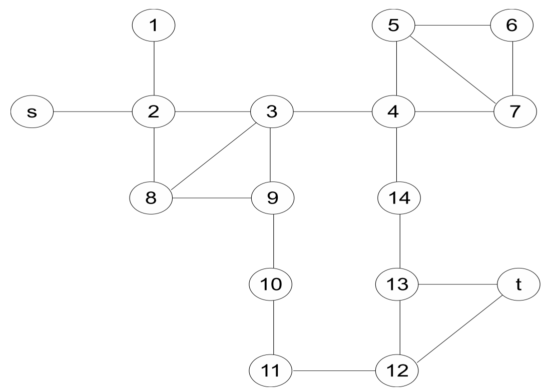


Figure 1: The example graph.

[For example, in the graph shown in Figure 1, $\tau = 10$, which corresponds to $P(6) : s \to 2 \to 3 \to 4 \to 7 \to 6 \to 7 \to 4 \to 14 \to 13 \to t$.]

C26. The diameter of a tree $T = (V, E)$ is given by $\max_{u,v \in V}\{\delta(u, v)\}$, where $\delta(u, v)$ is the shortest path distance (i.e., the length of the shortest path) between vertices u and v. So, the diameter is the largest of all shortest path distances in the tree.

(a) Write pseudo-code for an efficient algorithm to compute the diameter of a given tree T.

(b) Analyze the time complexity of your algorithm.

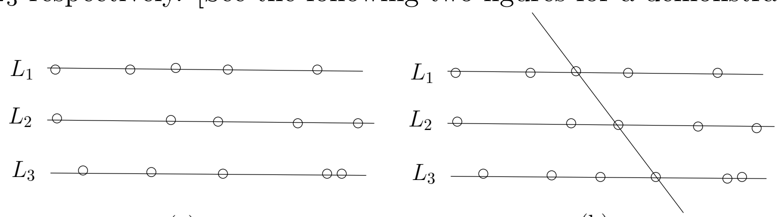
(c) What is its space complexity?

(d) Clearly mention the data structure(s) used by your algorithm.

(e) A vertex c is called a center of a tree T if the distance from c to its most distant vertex is the minimum among all vertices in V. Write an algorithm to determine and report a center of the given tree T.

C27. Consider three parallel lines L_1, L_2 and L_3. On each line L_i, a set of n points $\{p_{i1}, p_{i2}, \ldots, p_{in}\}$ is placed.

The objective is to identify a triplet of indices (k, ℓ, m) (if exists) such that a straight line can pass through p_{1k}, $p_{2\ell}$ and p_{3m} on L_1, L_2 and L_3 respectively. [See the following two figures for a demonstration.]



(a) (b)

In Figure (a), there does not exist any triplet (k, ℓ, m) such that a straight line can pass through $p_{1k}, p_{2\ell}$ and p_{3m}. In Figure (b), the triplet $(3, 3, 4)$ is a solution since a straight line passes through p_{13}, p_{23} and p_{34}.

Present an efficient algorithm for solving this problem. Justify its correctness and worst case time complexity.

[Full credit will be given if your algorithm is correct, and of worst case time complexity $O(n^2)$.]

C28. Let $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2} + 1$ for $n > 2$.

(a) Express 63 as a sum of distinct a_i's.

(b) Write an algorithm to express any positive integer k as a sum of at most $\lceil \log_2 k \rceil$ many distinct a_i's.

(c) Prove the correctness of your algorithm.

C29. (a) How many distinct labeled spanning trees does a complete graph of n vertices have? Give a formal argument to establish your answer.

(b) A least expensive connection route among n houses needs to be designed for a cable-TV network. Consider the following algorithm $\mathcal{A}$ for finding a spanning tree.

Algorithm $\mathcal{A}$
Input: $G = (V, E)$
Output: Set of edges $M \subseteq E$
Sort E in decreasing order of cost of edges;
$i \leftarrow 0$;
while $i < |E|$ **do**

begin
  Let $temp = (u_1, u_2)$ be the i-th edge $E[i]$ in E;
  Delete $E[i]$, i.e., replace $E[i]$ by ϕ;
  if u_1 is disconnected from u_2 **then**
    restore $temp$ in list E as $E[i]$;
  $i \leftarrow i + 1$;
end
return the edges in E which are not ϕ;

i. Prove that the algorithm $\mathcal{A}$ can be used to correctly find a least cost connection route, given a set of n houses and information about the cost of connecting any pair of houses.
ii. What is the worst case time complexity of $\mathcal{A}$?
iii. If all the edges have distinct cost, how many solutions can there be?

C30. (a) Let $T = (V, E)$ be a tree, and let $v \in V$ be any vertex of T.

- The *eccentricity* of v is the maximum distance from v to any other vertex in T.
- The *centre* C of T is the set of vertices which have the minimum eccentricity among all vertices in T.
- The *weight* of v is the number of vertices in the largest subtree of v.
- The *centroid* G of T is the set of vertices with the minimum weight among all vertices in T.

Construct a tree T that has disjoint centre and centroid, each having two vertices, i.e., $C \cap G = \varnothing$ and $|C| = |G| = 2$.

(b) A vertex cover of a graph $G = (V, E)$ is a set of vertices $V' \subseteq V$ such that for any edge $(u, v) \in E$, either u or v (or both) is in V'. Write an efficient algorithm to find the minimum vertex cover of a given tree T. Establish its correctness. Analyse its time complexity.

C31. You are given k sorted lists, each containing m integers in ascending order. Assume that (i) the lists are stored as singly-linked lists with one integer in each node, and (ii) the head pointers of these lists are stored in an array.

(a) Write an efficient algorithm that merges these k sorted lists into a single sorted list using $\Theta(k)$ additional storage.

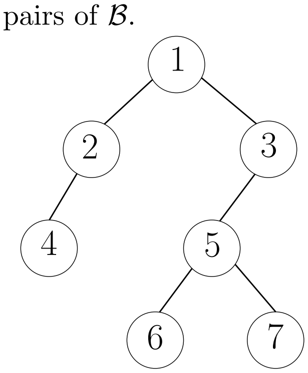
(b) Next, write an efficient algorithm that merges these k sorted lists into a single sorted list using $\Theta(1)$ additional storage.

(c) Analyse the time complexity of your algorithm for each of the above two cases.

C32. (a) Assume you have a chocolate bar containing a number of small identical squares arranged in a rectangular pattern. Our job is to split the bar into small squares by breaking along the lines between the squares. We obviously want to do it with the minimum number of breakings. How many breakings will it take?

(b) Consider that the chocolate bar has n breaking lines along the length and m breaking lines along the breadth. Write a C function that will take n, m as inputs and print the line numbers along the length and the breadth according to your strategy of breaking the chocolate.

C33. (a) Let $\mathcal{B}$ be a rooted binary tree of n nodes. Two nodes of $\mathcal{B}$ are said to be a sibling pair if they are the children of the same parent. For example, given the binary tree in Figure **??**, the sibling pairs are (2, 3) and (6, 7). Design an $O(n)$ time algorithm that prints all the sibling pairs of $\mathcal{B}$.



(b) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complete binary trees that are heaps as well. Assume $\mathcal{H}_1$ and $\mathcal{H}_2$ are max-heaps, each of size n. Design and analyze an efficient algorithm to merge $\mathcal{H}_1$ and $\mathcal{H}_2$ to a new max-heap $\mathcal{H}$ of size $2n$.

C34. (a) Let A and B be two arrays, each containing n distinct integers. Each of them is sorted in increasing order. Let $C = A \cup B$. Design an algorithm for computing the median of C as efficiently as you can.

(b) Let $G = (V, E)$ be an undirected weighted graph with all edge weights being positive. Design an efficient algorithm to find the *maximum* spanning tree of G.

C35. (a) Construct a deterministic finite automaton accepting the following language:

$\{w \in \{0,1\}^* : w$ has an equal number of 01's and 10's $\}$.

For example, 101 is in the language because it contains one instance of 10 and one instance of 01 as well.

(b) Consider the following statement:

For all languages $L \subseteq \{0,1\}^*$, if L^* is regular then L is regular.

Is the above statement true? Justify your answer.

C36. (a) The average memory access time for a microprocessor with first level cache is 3 clock cycles.

- If data is present in the cache, it is found in 1 clock cycle.
- If data is not found in the cache, 100 clock cycles are needed to get it from off-chip memory.

It is desired to obtain a 50% improvement in average memory access time by adding a second level cache.

- This second level cache can be accessed in 6 clock cycles.
- The addition of this second level cache does not affect the first level cache.
- Off-chip memory accesses still require 100 clock cycles.

To obtain the desired speedup, how often must data be found in the second level cache?

(b) Two modules M_1 and M_2 of an old machine are being replaced by their improved versions M_3 and M_4, respectively in a new machine. With respect to the old machine, the speed-up of these modules (M_3 and M_4) are 30 and 20, respectively. Only one module is usable at any instant of time. A program P, when run on the old machine, uses M_1 and M_2 for 30% and 20% of the total execution time, respectively. Calculate the overall speed-up of P when it is executed on the new machine.

C37. (a) Two queries equivalent to each other are specified for a relation $R(A, B, C, D, E, F)$. The queries are:

- $\pi_{A,B,C}(\sigma_{B>500}(R))$

- $\sigma_{B>500}(\pi_{A,B,C}(R))$

The system maintains a $B+$ tree index for (A, B, C) on R. However, the index is unclustered. The relation R occupies 100 pages and the index structure needs 5 pages only. Compute the number of disk accesses required for each of the queries and thereby decide which one of the two queries will be preferred by the query optimizer for minimum cost of execution. The cost of query execution is primarily dependent on the number of disk accesses.

(b) In a LAN, n^2 routers are connected in an $n \times n$ mesh such that $R(i, j)$ represents a router in the i-th row and j-th column of the mesh.

(i) Find how many distinct shortest paths exist between two routers $R(i_1, j_1)$ and $R(i_2, j_2)$ $(1 \leq i_1, j_1, i_2, j_2 \leq n)$. Two paths are distinct if they differ in at least one link.

(ii) At most how many of these distinct shortest paths will be node disjoint, i.e., with no common node except the source and the destination? Justify your answer.

C38. (a) Consider a uniprocessor system with four processes having the following arrival and burst times:

	Arrival Time	CPU Burst Time
P1	0	10
P2	1	3
P3	2.1	2
P4	3.1	1

(i) Calculate the average waiting time and also the average turnaround time if shortest (remaining) job first (SJF) scheduling policy is used with pre-emption. Assume that the context switching time is zero. Note that in SJF, if at any point there is a tie, then the job that arrived earlier will get priority.

(ii) Now consider the continuous arrival of new jobs at times 4, 5, 6, 7, ... following $P4$, with CPU burst times of 2 units each. In this case, what will be the turnaround time of $P1$? Justify your answer.

(b) A heavily loaded 1 km long, 10 Mbps token ring network has a propagation speed of 200 meter per micro-second. Fifty stations are uniformly spaced around the ring. Each data packet is 256 bits long, including 32 bits of header. The token is of 8 bits. What is the effective data rate of the network assuming the stations always have packets to transmit?