2016

BOOKLET NO. Test Code: CSB

Afternoon Time: 2 hours

On the answer-booklet, write your Registration Number, Test Code and Number of this booklet in appropriate places.

ATTENTION!

Read the following carefully before you start.

The question paper is divided into the following two groups:

Group A: Total of 30 marks. Attempt ALL QUESTIONS.

Group B: Total of 70 marks. It has five sections which are:

I. Computer Science

II. Electrical and Electronics Engineering

III. Mathematics

IV. Physics

V. Statistics

Select ONLY ONE SECTION, and answer ANY FIVE QUESTIONS from the selected section.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/ OR ON THE ANSWER BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATING/COMMUNICATING DEVICES OR MATHEMATICAL TABLES.

STOP! WAIT FOR THE SIGNAL TO START!

GROUP A

Answer all questions

A1. A standard deck of cards, containing 13 cards in each of 4 suites, is distributed equally among 4 players.

(a) Show that each player must have at least 4 cards of the same suite. [5]

(b) Define data structures to represent (i) the deck of cards, and (ii) a distribution of the cards to the four players. [5]

(c) Using your data structures, write an algorithm which distributes the deck of cards one by one to the four players in a cyclical manner. [5]

A2. Consider polynomials of degree n of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

where a_i's are integers in $[-20, 20]$.

(a) How many such polynomials $p(x)$ are there? [3]

(b) Show how to use an array to represent such a polynomial. [4]

(c) Given such a polynomial $p(x)$, write an algorithm to find all the integer roots of $p(x)$ that are in $[0, 100]$. [4]

(d) How many polynomials $p(x)$ are there such that $a_i < a_{i+1}$, for $i = 0, 1, \ldots, n-1$? [4]

GROUP B

I. COMPUTER SCIENCE

Answer any five questions

C1. (a) Fill in the blanks in the following routine that computes the preorder traversal of a binary tree using a stack. The blanks should be filled in using only the standard `push()` or `pop()` operations with appropriate arguments. [4]

```
void Stack_Preorder (Tree T, Stack S) {
  if (T == NULL) return; else ___________;
  while (!isempty(S)) {
    T = ___________;
    print_element(T -> Element);
    if (T -> Right != NULL) ___________;
    if (T -> Left != NULL) ___________;
  }
}
```

(b) What will be the output if the following C program is executed on a little endian machine? Assume `sizeof(int)` and `sizeof(char)` are 4 and 1 respectively. [4]

```
#include <stdio.h>
int main(void) {
  int i;
  int n = 261;
  char *ptr;
  ptr = (char *)&n;
  for(i = 0; i < 4; i++)
    printf("%d ", *ptr++);
  return 0;
}
```

(c) Let `root` be the root of a binary tree and n be an integer. The node data structure for a binary tree is defined as follows.

```
struct node{
    int data;
    struct node *left, *right;
};
```

Consider the function `X()` defined below.

```
int X(struct node *node, int a) {
  if (node == NULL)
    return (a == 0);
  else {
    int b = a - node->data;
    return (X(node->left, b) || X(node->right, b));
  }
}
```

If `X(root, n)` returns 1, what property must the binary tree have? Justify your answer. [6]

C2. You are given a set $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of n (≥ 1) points on a plane, where for all i, $1 \leq i \leq n$, $x_i \geq 0$, $y_i \geq 0$. A point $(x_i, y_i) \in S$ is said to *dominate* a point $(x_j, y_j) \in S$ if $x_i > x_j$ and $y_i > y_j$. A point in S is called *maximal* if no point in S dominates it.

(a) Design an algorithm to find *any one* maximal point of S in $O(n)$ time. [4]

(b) Design an algorithm to find *all* maximal points of S in $O(n \log n)$ time. [10]

C3. Let T be a binary tree (not necessarily balanced) with n nodes. A path $v_0, v_1, \cdots, v_{k-1}, v_k$ in T is said to be a *maximal path* if it satisfies the following three conditions:

- the interior vertices $v_1, \cdots, v_{k-1}$ are all of degree 2,
- v_0 is either the root or a degree 3 vertex, and
- v_k is either a leaf or a degree 3 vertex.

Let the *Prune*() operation be defined as follows:

Step 1. For every maximal path $v_0, v_1, \cdots, v_{k-1}, v_k$ in T, remove the interior vertices $v_1, \cdots, v_{k-1}$, and add an edge between v_0 and v_k.

Step 2. Remove every leaf of T.

Based on the above definition,

(a) show that, after a *Prune*() operation, the tree will contain no more than $n/2$ nodes; [10]

(b) hence, or otherwise, show that after successively applying the *Prune*() operation on T for $O(\log n)$ iterations, T will be reduced to a tree with a single node. [4]

C4. Let $L = \{0^p\, 1^q\, 2^{\max(p,q)} \mid p,\ q \text{ are positive integers}\}$. Is L regular? Is L context-free? Justify your answer in each case. [6 + 8]

C5. (a) Prove or disprove: Every Eulerian bipartite graph has an even number of edges. [4]

(b) Fibonacci numbers are defined as follows: $F_1 = 1$, $F_2 = 1$ and $F_i = F_{i-1} + F_{i-2}$ for $i = 3, 4, \ldots$. Prove that every positive integer n (≥ 3) can be written as a sum of at most $\log_2 n$ distinct Fibonacci numbers. [10]

C6. (a) For a computer C, the cycles per instruction (CPI) is 1.0. The miss rate and miss penalty are 2% and 25 clock cycles, respectively. Load and store instructions account for 50% of the program. Compute the speed-up if C's miss rate is reduced to 0%. [4]

(b) You are given a positive integer A, stored in 32-bit 2's complement form in register t_1. Write a sequence of machine-language instructions to compute $N = \lfloor 31 * A/4 \rfloor * 17$, where $*$, $/$, $\lfloor \ \rfloor$ represent multiplication, division, and floor operations, respectively. Note that the concerned machine has ten 32-bit general-purpose registers $t_1, t_2, \cdots, t_{10}$, and it only supports the following machine-level instructions involving integers:

ADD t_i, t_j, t_k	adds the contents of registers t_j and t_k; stores the result in t_i
SUB t_i, t_j, t_k	subtracts the content of register t_k from that of register t_j; stores the result in t_i

In addition, the machine also supports the following bit-level operations:

SLL t_i, t_j, d	shifts the bit-pattern in register t_j by d bits to the left and stores the result in t_i.
SRL t_i, t_j, d	shifts the bit-pattern in register t_j by d bits to the right and stores the result in t_i.

For the above operations, d is an integer less than or equal to 32. While shifting the bits in t_j, the empty bits from the right-end/left-end are filled with 0's.

The value of N should be finally saved in register t_1. Justify your reasoning when you write the code. Assume that N does not exceed $2^{31} - 1$. Note that the machine *does not support* any other instructions (e.g., multiply or divide).

[10]

C7. (a) Two computers A and B are connected over a circuit-switched network. All links in the network use time division multiplexing (TDM) with 24 slots/sec and have a bit rate of 1.536 Mbps. It takes 500 msec to establish an end-to-end circuit between A and B. How long does it take to send a file of $640,000$ bits from A to B? [5]

(b) Consider the following extract from a program, written in a C-like language, that computes the transpose of a matrix.

```
for (i = 0; i < N; i++)
    for (j = 0; j < N; j++)
        B[i,j] = A[j,i];
```

`A` and `B` are `N`×`N` matrices with floating point entries that are stored in memory in row-major order as shown in the example below.

A[0,0]	A[0,1]	A[0,2]	...	A[0,N-1]	A[1,0]	A[1,1]	...	A[N-1,N-1]

This program runs under an operating system that uses paging based memory management. Considering **only** memory references to the matrix entries, and using the information given below, compute the page fault rate for the matrix transposition code given above.

- Page size: 2^{10} bytes
- Number of frames allocated to the program: 8
- Page replacement policy: LRU
- `N` $= 2^{16}$
- Size of each matrix entry: 8 bytes
- Each of `A` and `B` is stored starting from the beginning of a page.
- None of the pages allocated to `A` or `B` are initially in memory. [9]

C8. (a) Consider a relational schema $X(P, Q, R, S, T, U)$ on which the following functional dependencies hold:

$$\{P \rightarrow Q,\ QR \rightarrow S,\ T \rightarrow R,\ S \rightarrow P\}.$$

Are PTU and QRU candidate keys? Justify your answer. [4]

(b) Let $R(A, B, C)$ and $S(B, C, D)$ be two relations. Consider the following relational algebra expressions.

- E_1: $\pi_{A,B}(\sigma_{C=c}(R) \bowtie S)$
- E_2: $\pi_{A,B}(\sigma_{C=c}(R)) \bowtie \pi_B(\sigma_{C=c}(S))$
- E_3: $\sigma_{B=b}(\sigma_{A=a \wedge C=c}(R) \bowtie \sigma_{D=d}(S))$
- E_4: $\sigma_{A=a \wedge B=b \wedge C=c}(R) \bowtie \sigma_{B=b \wedge C=c \wedge D=d}(S)$

In the above expressions, a, b, c and d are specific values that the attributes A, B, C and D may respectively take. Here, σ and π denote the standard selection and projection operations respectively.

(i) Show that E_1 and E_2 will produce the same result.

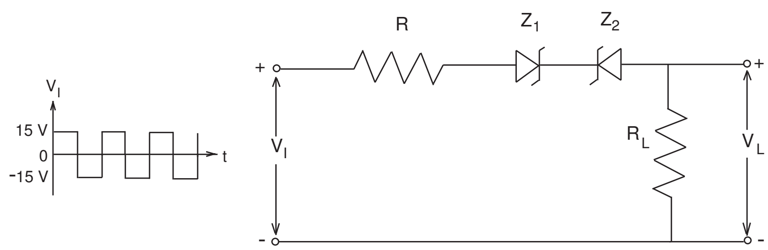
(ii) Show that E_3 and E_4 will produce the same result.

(iii) From the query optimization point of view, which among E_3 and E_4 would you prefer and why? [4 + 3 + 3]

II. ELECTRICAL AND ELECTRONICS ENGINEERING

Answer any five questions

E1. (a) Consider the following circuit, where two Zener diodes Z_1 and Z_2 have negligible forward voltage drops. The constant reverse breakdown voltages of Z_1 and Z_2 are $10V$ and $5V$ respectively. If the input voltage V_I is a $15V$ square wave and $R = R_L = 12\Omega$, plot the output voltage V_L. Justify your answer. [4]



(b) Explain what would happen if the Zener diodes are replaced by ideal regular diodes. [10]

E2. Consider an alphabet of 6 symbols whose probabilities in a particular text are as follows:

A	B	C	D	E	F
0.1	0.05	0.1	0.05	0.4	0.3

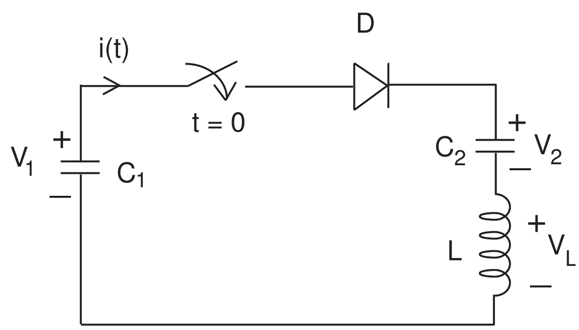
Consider the following game. From the given text, a symbol is picked up randomly by your opponent and you need to determine which symbol it is, by asking 'yes/no' questions pertaining to individual symbols (for example, "is it A?") that will be truthfully answered.

(a) What will be your strategy so that "the average number of questions to be asked to determine a symbol" is minimized? Justify your answer. [5]

(b) With your strategy, what is the average number of questions that you need to ask in order to determine a symbol for the above data? [3]

(c) How can your strategy be exploited in generating decodable binary codes for the above set of alphabets? What do the code-words optimize? [6]

E3. (a) Consider the following source-free circuit, where D is an ideal diode and C_1, C_2 and L are ideal circuit elements. Plot $i(t)$ assuming $V_1(0) = V_0$ and $V_2(0) = 0$, as long as D conducts. [10]



(b) Without making further calculations, outline what would happen if a resistance R is introduced in series in the circuit. [4]

E4. (a) A 3-phase, 50 Hz, Δ-Y connected transformer supplies power to a Y-connected balanced load. Let its maximum efficiency be achieved at $\frac{3}{4}$ full-load. Determine the full-load efficiency of the transformer at power factor 0.8, when the input line voltage is 2.2 kV and the output phase current is 10A. Assume a turns ratio of 1:5 and the iron loss to be 3.6 kW. [6]

(b) A 12-pole, 3-phase alternator driven at a speed of 500 $r.p.m.$ supplies $200\sqrt{3}$ V to an 8-pole, 3-phase, Y-connected induction motor. The rotor is also Y-connected with stator-to-rotor turns ratio of 2:1. Assume that the rotor resistance per phase is 0.6 Ω and the standstill rotor reactance per phase to be 0.8 Ω.

(i) Calculate the speed of the rotor if slip is 4%.

(ii) Calculate the rotor current.

(iii) If the rotor speed decreases to 705 $r.p.m.$ due to extra load, calculate the frequency of the rotor e.m.f.

[2 + 4 + 2]

E5. (a) Consider a Boolean function $F(x_1, x_2, x_3, x_4, x_5)$ such that $F = 1$ if exactly three of its inputs have the value 1.

(i) How many essential prime implicants does F have?

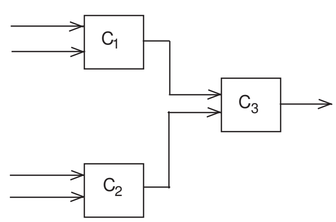
(ii) What is the total number of true minterms in F?

[3 + 3]

(b) Realize the Boolean function $F(x_1, x_2, x_3, x_4)$ represented by the Karnaugh map given below with the circuit structure shown alongside. Each of C_1, C_2 and C_3 is a two-input gate where $C_i \in \{AND, NAND, NOR, XOR, XNOR\}$, $i = 1, 2, 3$. You may use any number of additional inverters as needed.

[8]

X_4X_3 \ X_2X_1				
		1		1
	1	1		
	1			1



E6. (a) Design a synchronous sequential circuit C with one binary input x and one binary output z such that $z = 1$, if and only if any two of the last three inputs (including the present one) were 1. You will be given D-flipflops, logic gates and a clock source as needed. [6]

(b) Draw the state diagram for the above-mentioned synchronous sequential circuit C. [4]

(c) The circuit C receives an input stream (leftmost bit arriving first at C) as follows:

$$\text{Input}(x) : 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0$$

$$------ > \text{time}$$

Determine the corresponding output sequence in accordance with the design of C. [4]

E7. (a) Consider a linear time-invariant analog filter with an impulse response

$$h(t) = \begin{cases} e^{0.2t} + e^{0.6t} & \text{for } t > 0; \\ 0 & \text{otherwise.} \end{cases}$$

(i) Determine the transfer function of the system.

(ii) $h(t)$ is now sampled with a sampling frequency of 1 Hz to produce a discrete-time impulse response $h[n]$. Determine the corresponding transfer function.

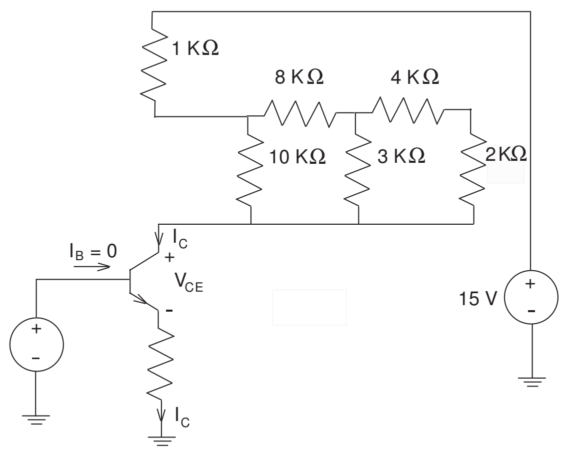
(iii) Does $h[n]$ represent a stable discrete-time system? Justify your answer. [3 + 4 + 3]

(b) Determine the causal sequence whose Z-transform is

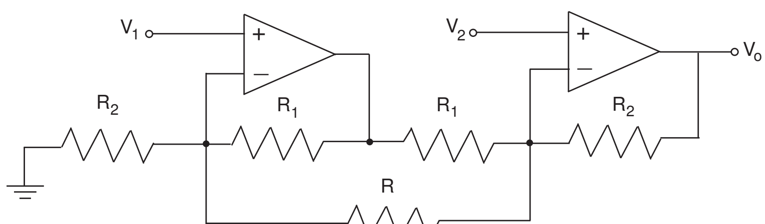
$$\frac{1}{1 - 0.7z^{-7}}$$

[4]

E8. (a) For the transistor circuit given below, determine V_{CE} if $I_C = 1\ mA$. [7]



(b) For the circuit in the figure below, derive an expression for $\frac{V_O}{(V_1 - V_2)}$. [7]



III. MATHEMATICS

Answer any five questions

M1. (a) Find the number of positive integers with k digits, that do not contain the digit 7. [6]

(b) Let S denote the set of all positive integers that do not contain the digit 7. Prove that the infinite series

$$\sum_{n \in S} \frac{1}{n}$$

converges. [8]

M2. (a) Let x be an irrational number. Show that for any positive integer $k \geq 1$, there exists $\delta_k > 0$, such that the open interval $(x - \delta_k, x + \delta_k)$ does not contain any rational number of the form $\frac{p}{k}$, where p is an integer. [6]

(b) Consider the function f defined on $[1, 2]$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in [1,2] \text{ is irrational} \\ \frac{1}{k} & \text{if } x \in [1,2] \text{ is a rational } \frac{p}{k} \text{ where } p \text{ and} \\ & k \text{ are integers prime to each other} \end{cases}$$

Show that f is discontinuous at every rational $x \in [1, 2]$ and continuous at every irrational $x \in [1, 2]$. [8]

M3. Let S be a ring with the property that for all $a \in S$ with $a \neq 0$, there exists a unique $b \in S$ such that $aba = a$.

(a) Consider an element $a \in S$ with $a \neq 0$. Show that (i) $ac = 0$ implies $c = 0$ and (ii) $ca = 0$ implies $c = 0$. [3 + 2]

(b) Show that for every $a \in S$ with $a \neq 0$, there is a unique $b \in S$ such that $ba = ab = 1$. [9]

M4. $\mathbf{A}_{n \times n}$ is said to be a *general* positive definite matrix if $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all non-zero $\mathbf{x} \in \mathbb{R}^n$.

(a) If $\mathbf{A}$ is a general positive definite matrix, would $\mathbf{A} + \mathbf{A}'$ have all eigenvalues positive? Justify your answer. [7]

(b) Give an example of a matrix $\mathbf{A}$ which has all eigenvalues positive, but is not a general positive definite matrix. [7]

M5. Let $\mathbf{A} = ((a_{ij}))$ be an $n \times n$ matrix such that $a_{ij} = 1$ if $|i - j| = 1$ and $a_{ij} = 0$ if $|i - j| > 1$.

(a) Show that the (i, j)-th element of $\mathbf{A}^2$ is 0 if $|i - j| > 2$. [5]

(b) Show that the characteristic polynomial of the matrix $\mathbf{A}$ is the polynomial $p(\cdot)$ of the smallest degree such that $p(\mathbf{A})$ is a matrix with all entries 0. [9]

M6. Let S_n be the group consisting of all permutations of n symbols.

(a) Let $n = n_1 + \cdots + n_k$, where $k \geq 2$. Show that S_n has a cyclic subgroup of order $\mathrm{lcm}\,(n_1, \ldots, n_k)$. [4]

(b) Show that S_5 has a normal subgroup H such that S_5/H is of order 20. [4]

(c) Consider the following permutations:

$$\mathbf{\Pi_1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}$$

$$\mathbf{\Pi_2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 1 & 5 & 2 \end{pmatrix}$$

$$\mathbf{\Pi_3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$$

For each of the following pairs, determine whether they are conjugate and if so, obtain a conjugating permutation.

(i) $\mathbf{\Pi_1}$ and $\mathbf{\Pi_2}$

(ii) $\mathbf{\Pi_2}$ and $\mathbf{\Pi_3}$ [3 + 3]

M7. (a) Find all solutions in x (with $0 \leq x < 720$) to the following system of equations: [8]

$$x \equiv 1 \bmod 5$$

$$x \equiv 2 \bmod 8$$

$$x \equiv 3 \bmod 9$$

(b) Let n be a positive integer. Show that the number of partitions of n, having one part more than $\frac{n}{2}$, is equal to the number of partitions of n having more than $\frac{n}{2}$ parts. [6]

M8. (a) Give an example of a graph having two maximal independent sets with a non-empty intersection. [4]

(b) In a graph $G = (V, E)$, let S be a maximal independent set. Show that $V \setminus S$ is a minimal vertex cover. [4]

(c) Let I_1, I_2 be two maximal independent sets and $T = I_1 \cap I_2$. Show that for every vertex $u \in I_1 \setminus T$, there is a vertex $v \in I_2 \setminus T$ such that (u, v) is an edge of G. [6]

IV. PHYSICS

Answer any five questions

P1. (a) A point mass moves under a force directed towards a fixed point and the force depends on the distance from the fixed point.

(i) Show that the particle would be forced to move in a plane.

(ii) Show that the area traversed by the particle in unit time is constant.

(iii) If the force $\vec{F}$ varies as the inverse of the square of the distance, show that

$$\vec{\nabla} \times \vec{F} = 0$$

Discuss its implication on conservation of energy.

[2 + 2 + 3]

(b) A simple pendulum hangs from the ceiling of an elevator which is moving down with constant acceleration f.

(i) Write down the Lagrangian of the system.

(ii) Construct the Hamiltonian and obtain the equation of motion from the Hamiltonian.

(iii) Under what condition will the time period of the pendulum be infinite? [3 + 2 + 2]

P2. (a) In an inertial frame, two events have the space time coordinates $\{x_1, y, z, t_1\}$ and $\{x_2, y, z, t_2\}$.
Let $x_2 - x_1 = 2c(t_2 - t_1) > 0$. Consider another inertial frame which moves along x-axis with velocity u with respect to the first one. Find the value of u for which the two events are simultaneous in the latter frame (c represents the velocity of light in vacuum). [6]

(b) The transformation equation between two sets of coordinates $[Q, P]$ and $[q, p]$ are

$$Q = \ln(1 + \sqrt{q}\ \cos p), \quad P = 2(1 + \sqrt{q}\ \cos p)\sqrt{q}\ \sin p.$$

Show that if q, p are canonical variables, then Q, P are also canonical variables. [8]

P3. (a) Consider a simple harmonic oscillator in one dimension with the Hamiltonian

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

where a and $a^\dagger$ have the usual meaning of annihilation and creation operator. Show that $[a^\dagger, H] = -a^\dagger$. [6]

(b) The wave function of the Harmonic oscillator at $t = 0$ is given by

$$\psi(0) = N\,(2|0> + |3>)$$

where N is the normalization constant and $|n>$ is the eigenfunction of corresponding energy eigenvalue

$$E_n = \hbar\omega(n + \frac{1}{2}).$$

(i) Find the normalization constant N.

(ii) Calculate the expectation value of the energy for this wave function.

(iii) Find the wave function $\psi(t)$ at time t. [2 + 3 + 3]

P4. (a) A particle of mass m is confined to a line and has a wavefunction $\psi = C\exp(-a^2x^2/2)$. Calculate C in terms of a. Find the potential energy at a distance x from the origin if the total energy of the particle is $(h^2a^2)/(8\pi^2 m)$. [3 + 5]

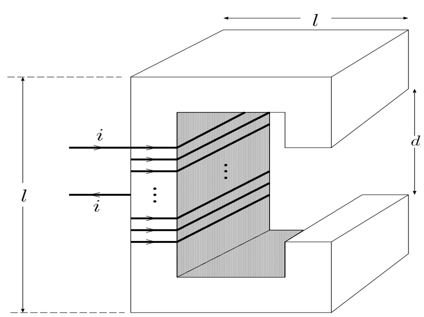
(b) What is the velocity of a relativistic electron of mass m whose kinetic energy is equal to the energy of a photon of wavelength λ. Consider c and h to be the velocity of light and Planck's constant, respectively. [6]

P5. (a) In a region there is a uniform electric field E and a uniform magnetic field B, both directed along the z-axis. A particle of mass m and charge Q is injected at time $t = 0$ with a velocity v_0 along the x-axis. Find the velocity of the particle at time t. [7]

(b) For the C-shaped magnet made of soft iron core as shown in the following figure, μ_R for soft iron is 3000, $i = 2A$, $l = 50$ cm and $d = 5$ cm. Let N be the number of turns.

(i) Find the $B-$field.

(ii) Determine the number of turns required to produce a field of 1000 Gauss. [5 + 2]

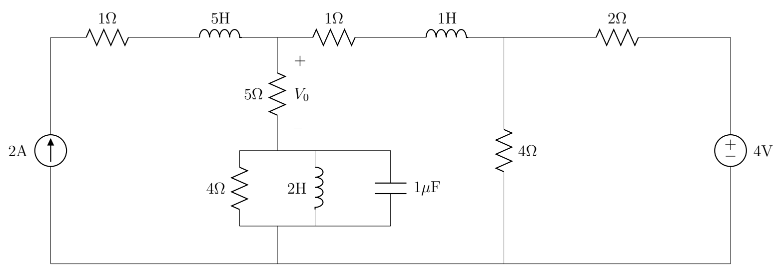


P6. (a) The cross-section for excitation upon impact with an electron beam for an atomic level X is $\sigma_X = 1.5X10^{-20}cm^2$. The level has a lifetime $\tau = 10^{-8}$sec. It decays to the level Y. Let an electron beam of 10 mA/cm^2 be allowed to pass through a vapour of these atoms at a pressure of 0.1 torr at room temperature. Calculate the number of atoms per unit volume at equilibrium for the energy level X. [7]

(b) One mole of a diatomic perfect gas which is initially at a temperature T_0, expands from volume V_0 to $2V_0$ (i) at constant temperature, (ii) at constant pressure. Calculate the work done and the heat absorbed in each case, given that $C_v = \frac{5}{2}R$ for diatomic gas. [3 + 4]

P7. (a) Find the RMS speed and the de Broglie wavelength of an Oxygen molecule of mass M at a temperature T where k is the Boltzmann constant. [5]

(b) Consider the following circuit. Calculate V_0 at steady state. [9]



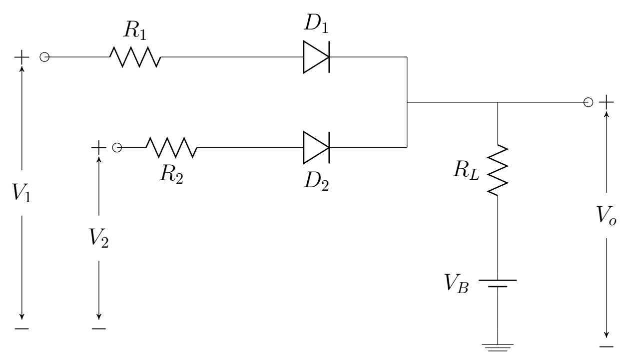
P8. (a) Consider the following circuit, where D_1 and D_2 are two ideal diodes. Find the output voltage V_0 for:

(i) $V_1 = V_2 = 10\ V$

(ii) $V_1 = 10\ V$ and $V_2 = 0\ V$

Consider that $R_1 = R_2 = 1\ \Omega$; $R_L = 9\ \Omega$; and $V_B = 5\ V$.

[3 + 3]



(b) Simplify the following Boolean function in product-of-sums form and draw the logic diagram using NOR gates:

$$F(A, B, C, D) = \Sigma(0, 1, 2, 5, 8, 9, 10).$$

[8]

V. STATISTICS

Answer any five questions

S1. (a) If m red balls and n green balls are arranged at random in a line, find the expected number of adjacent places occupied by two balls of different colours.

[For example, in case of 2 red balls and 2 green balls, the arrangement $RGRG$ will have 3 such adjacent places, while $RGGR$ will have 2 such adjacent places.] [6]

(b) Let X be the number of times a fair dice has to be rolled until all six faces appear at least once. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [8]

S2. (a) Let X, Y have a bivariate normal distribution with zero means, unit variances, and correlation coefficient ρ.

(i) Find the density of X/Y.

(ii) Find the probability $P(X > 0, Y > 0)$. [4 + 3]

(b) Let $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{pmatrix}$ be a d-variate standard normal vector, where $\mathbf{X}_i$ is of dimension d_i, for $i = 1, 2, 3$. For $j = 1, 2$, define $Z_j = \mathbf{X}_j^{'}\mathbf{X}_j/\mathbf{X}^{'}\mathbf{X}$. Find the joint distribution of (Z_1, Z_2). [7]

S3. A book is checked by successive proof-readers to identify and correct typographical errors. Assume that a proof-reader who receives a book containing i errors will return it with j errors, $0 \leq j \leq i$, with equal probability for each j. The book is then passed on to the next proof-reader. Assume that the book initially has 50 errors.

(a) Formulate the above process as a Markov chain, clearly describing the state space and the transition probabilities. [4]

(b) Show that it is an absorbing chain with only one absorbing state. [3]

(c) Find the expected number of proof-readers required till all 50 errors are corrected. [7]

S4. (a) Let $X_1, \ldots, X_n$ be independent and identically distributed $N(\mu, 1)$ variables. Find the uniformly minimum variance unbiased estimator for $\Phi(\mu)$, where Φ denotes the cumulative distribution function of the standard normal distribution. [8]

(b) Consider the uniform distribution on

$$S_\theta = \{\mathbf{x} = (x_1, x_2)';\ \theta \leq \|\mathbf{x}\| \leq \theta + 1\}.$$

If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are n independent observations from this distribution, find the maximum likelihood estimate of θ. [6]

S5. (a) Let $Y = X^Z$, where $X \sim U(0,1)$, $Z \sim N(0,1)$, with X and Z being independent. Find the best estimator (in terms of mean squared error) of Y based on X. [4]

(b) Consider a pair of hypotheses $H_0 : f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ and $H_1 : f(x) = \frac{1}{2}e^{-|x|}$.

(i) Construct a most powerful test of level α $(0 < \alpha < 1)$ based on a single observation.

(ii) If p_α is the power of this test, show that

$$\tan\left[\frac{\pi(1-p_\alpha)}{2}\right] + \log\,\alpha = 0$$

[7 + 3]

S6. (a) Consider a linear model $Y_i = \mathbf{x}_i'\boldsymbol{\beta} + \epsilon_i;\ i = 1, 2, \ldots, n$, where $\mathbf{x}_i$ and $\boldsymbol{\beta}$ are d-dimensional vectors $(d < n)$ and $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed as $N(0, \sigma^2)$. Let $\widehat{\boldsymbol{\beta}}$ be the least square estimator of $\boldsymbol{\beta}$. For any $\mathbf{m} \in \mathbb{R}^d$, show that $\mathbf{m}'\widehat{\boldsymbol{\beta}}$ is the best linear unbiased estimator of $\mathbf{m}'\boldsymbol{\beta}$. [7]

(b) Consider the linear models $Y_{jk} = \beta_{j0} + \sum_{i=1}^{d} \beta_{ji} x_{ik} + \epsilon_{jk}$ for $k = 1, 2, \ldots, n$ and $j = 1, 2, 3$, where the x_{ik}'s are non-stochastic and the ϵ_{jk}'s are independent and identically distributed with mean 0 and variance σ^2. Let $\hat{Y}_{jk}$ be the least square estimator of Y_{jk} for all j, k. If $Y_{1k} + Y_{2k} + Y_{3k} = 10$ for all k, does it necessarily imply $\hat{Y}_{1k} + \hat{Y}_{2k} + \hat{Y}_{3k} = 10$ for all k ? Justify your answer. [7]

S7. Consider the following linear model with three observations.

$$\begin{aligned} y_1 &= 5\mu + 3\theta_1 + 4\theta_2 + \theta_3 + \epsilon_1 \\ y_2 &= 2\mu + 2\theta_2 + \epsilon_2 \\ y_3 &= \mu - 3\theta_1 + 2\theta_2 - \theta_3 + \epsilon_3 \end{aligned}$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are independent and identically distributed as $N(0, \sigma^2)$. Assume that $\theta_1 + \theta_2 + \theta_3 = 1$.

(a) Show that $(\theta_1 - \theta_2)$ has a linear unbiased estimator. [3]

(b) Find the best linear unbiased estimator of $(\theta_1 - \theta_2)$. [3]

(c) Derive the standard F-statistic for testing $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$ for the above model. [8]

S8. (a) Let $\mathbf{X} = (X_1, X_2)'$ follow a bivariate normal distribution with mean vector $\mathbf{0} = (0, 0)'$ and dispersion matrix $\mathbf{I}_2$, the 2×2 identity matrix. Find the correlation coefficient between $Y_1 = X_1 \operatorname{sgn}(X_2)$ and $Y_2 = X_2 \operatorname{sgn}(X_1)$, where $\operatorname{sgn}(z) = 1, -1$, or 0 according as $z > 0, z < 0$, or $z = 0$. [6]

(b) Consider a classification problem involving two classes C_1 and C_2 with equal priors. Let C_1 be an equal mixture of $N((1,1)', \mathbf{I}_2)$ and $N((-1,-1)', \mathbf{I}_2)$, and, C_2 be that of $N((1,-1)', \mathbf{I}_2)$ and $N((-1,1)', \mathbf{I}_2)$. Show that probability of misclassification for any classifier is at least $2\Phi(1)\Phi(-1)$, where Φ denotes the cumulative distribution function of the standard normal distribution. [8]