2014

BOOKLET NO. Test Code: CSB

Afternoon Time: 2 hours

On the answer-booklet, write your Registration Number, Test Code and Number of this booklet in the appropriate places.

ATTENTION!

Read the following carefully before answering the test.

The question paper is divided into the following two groups:

Group A: Total of 20 marks. Attempt ALL questions.

Group B: Total of 80 marks. It has five sections. Select only one section, and answer any four questions from the selected section.

The five sections are: (i) Computer Science, (ii) Electrical and Electronics Engineering, (iii) Mathematics, (iv) Physics, and (v) Statistics.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/ OR ON THE ANSWER BOOKLET. YOU ARE NOT ALLOWED TO USE ANY CALCULATING/COMMUNICATING DEVICES OR MATHEMATICAL TABLES.

STOP! WAIT FOR THE SIGNAL TO START!

GROUP A

Answer all questions

A1. (a) Let $x = (x_1, x_2, \cdots, x_n) \in \{0,1\}^n$. By $H(x)$ we mean the number of 1's in $(x_1, x_2, \cdots, x_n)$.

Prove that $H(x) = \frac{1}{2}\left(n - \sum_{i=1}^{n}(-1)^{x_i}\right)$.

(b) Let x, y be two non-negative integers $< 2^{32}$. By $x \wedge y$ we mean the integer represented by the bitwise logical AND of the 32-bit binary representations of x and y. For example, if $x = 13$ and $y = 6$, then $x \wedge y$ is the bitwise AND of $0^{28}1101$ and $0^{28}0110$, resulting in $0^{28}0100$, which is 4 in decimal. (Here $0^{28}1101$ means twenty-eight 0's followed by the 4-bit string 1101.) Now consider the following pseudo-code:

integer $x, n = 0$;
while $(x \neq 0)\{$
  $x \leftarrow x \wedge (x - 1)$;
  $n \leftarrow n + 1$;
$\}$
print n**;**

(i) What will be the output of the pseudo-code for the input $x = 13$?

(ii) What will be the output of the pseudo-code for an arbitrary non-negative integer $x < 2^{32}$?

[$5 + (2 + 3) = 10$]

A2. Let A be a 30×40 matrix having 500 non-zero entries. For $1 \leq i \leq 30$, let r_i be the number of non-zero entries in the i-th row, and for $1 \leq j \leq 40$, let m_j be the number of non-zero entries in the j-th column.

(a) Show that there is a k such that $1 \leq k \leq 30$, $r_k \geq 17$ and there is an ℓ such that $1 \leq l \leq 40$, $m_\ell \leq 12$.

(b) Suppose that it is desired to create a stack of size 30 containing the values $r_1, \ldots, r_{30}$, not necessarily in order such that the top of the stack contains the value $\max_{1 \leq i \leq 30} r_i$. Write pseudo-code for creating such a stack using a single scan of the matrix A.

[4 + 6 = 10]

GROUP B

(i) COMPUTER SCIENCE

C1. (a) Assume you have a chocolate bar containing a number of small identical squares arranged in a rectangular pattern. Our job is to split the bar into small squares by breaking along the lines between the squares. We obviously want to do it with the minimum number of breakings. How many breakings will it take?

(b) Consider that the chocolate bar has n breaking lines along the length and m breaking lines along the breadth. Write a C function that will take n, m as inputs and print the line numbers along the length and the breadth according to your strategy of breaking the chocolate.

$[10 + 10 = 20]$

C2. (a) Let $\mathcal{B}$ be a rooted binary tree of n nodes. Two nodes of $\mathcal{B}$ are said to be a sibling pair if they are the children of the same parent. For example, given the binary tree in Figure 1, the sibling pairs are $(2,\ 3)$ and $(6,\ 7)$. Design an $O(n)$ time algorithm that prints all the sibling pairs of $\mathcal{B}$.

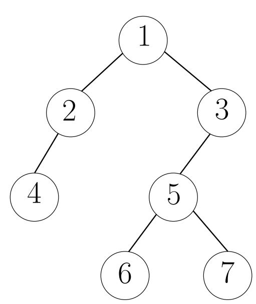


Figure 1: A rooted binary tree $\mathcal{B}$.

(b) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complete binary trees that are heaps as well. Assume $\mathcal{H}_1$ and $\mathcal{H}_2$ are max-heaps, each of size n. Design and analyze an efficient algorithm to merge $\mathcal{H}_1$ and $\mathcal{H}_2$ to a new max-heap $\mathcal{H}$ of size $2n$.

[$10 + 10 = 20$]

C3. (a) Let A and B be two arrays, each containing n distinct integers. Each of them is sorted in increasing order. Let $C = A \cup B$. Design an algorithm for computing the median of C as efficiently as you can.

(b) Let $G = (V, E)$ be an undirected weighted graph with all edge weights being positive. Design an efficient algorithm to find the *maximum* spanning tree of G.

[$10 + 10 = 20$]

C4. (a) Construct a deterministic finite automaton accepting the following language:

$\{w \in \{0,1\}^* : w$ has an equal number of 01's and 10's $\}$.

For example, 101 is in the language because it contains one instance of 10 and one instance of 01 as well.

(b) Consider the following statement:

For all languages $L \subseteq \{0,1\}^*$, if L^* is regular then L is regular.

Is the above statement true? Justify your answer.

[$10 + 10 = 20$]

C5. (a) The average memory access time for a microprocessor with first level cache is 3 clock cycles.

- If data is present in the cache, it is found in 1 clock cycle.
- If data is not found in the cache, 100 clock cycles are needed to get it from off-chip memory.

It is desired to obtain a 50% improvement in average memory access time by adding a second level cache.

- This second level cache can be accessed in 6 clock cycles.
- The addition of this second level cache does not affect the first level cache.
- Off-chip memory accesses still require 100 clock cycles.

To obtain the desired speedup, how often must data be found in the second level cache?

(b) Two modules M_1 and M_2 of an old machine are being replaced by their improved versions M_3 and M_4, respectively in a new machine. With respect to the old machine, the speed-up of these modules (M_3 and M_4) are 30 and 20, respectively. Only one module is usable at any instant of time. A program P, when run on the old machine, uses M_1 and M_2 for 30% and 20% of the total execution time, respectively. Calculate the overall speed-up of P when it is executed on the new machine.

$[12 + 8 = 20]$

C6. (a) Two queries equivalent to each other are specified for a relation $R(A, B, C, D, E, F)$. The queries are:

- $\pi_{A,B,C}(\sigma_{B>500}(R))$
- $\sigma_{B>500}(\pi_{A,B,C}(R))$

The system maintains a $B+$ tree index for (A, B, C) on R. However, the index is unclustered. The relation R occupies 100 pages and the index structure needs 5 pages only. Compute the number of disk accesses required for each of the queries and thereby decide which one of the two queries will be preferred by the query optimizer for minimum cost of execution. The cost of query execution is primarily dependent on the number of disk accesses.

(b) In a LAN, n^2 routers are connected in an $n \times n$ mesh such that $R(i, j)$ represents a router in the i-th row and j-th column of the mesh.

(i) Find how many distinct shortest paths exist between two routers $R(i_1, j_1)$ and $R(i_2, j_2)$ $(1 \leq i_1, j_1, i_2, j_2 \leq n)$. Two paths are distinct if they differ in at least one link.

(ii) At most how many of these distinct shortest paths will be node disjoint, i.e., with no common node except the source and the destination? Justify your answer.

$[10 + (6 + 4) = 20]$

C7. (a) Consider a uniprocessor system with four processes having the following arrival and burst times:

	Arrival Time	CPU Burst Time
P1	0	10
P2	1	3
P3	2.1	2
P4	3.1	1

(i) Calculate the average waiting time and also the average turnaround time if shortest (remaining) job first (SJF) scheduling policy is used with pre-emption. Assume that the context switching time is zero. Note that in SJF, if at any point there is a tie, then the job that arrived earlier will get priority.

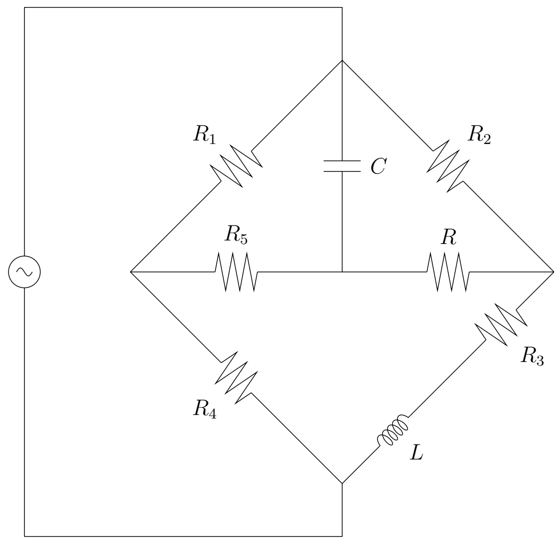
(ii) Now consider the continuous arrival of new jobs at times 4, 5, 6, 7, ... following $P4$, with CPU burst times of 2 units each. In this case, what will be the turnaround time of $P1$? Justify your answer.

(b) A heavily loaded 1 km long, 10 Mbps token ring network has a propagation speed of 200 meter per micro-second. Fifty stations are uniformly spaced around the ring. Each data packet is 256 bits long, including 32 bits of header. The token is of 8 bits. What is the effective data rate of the network assuming the stations always have packets to transmit?

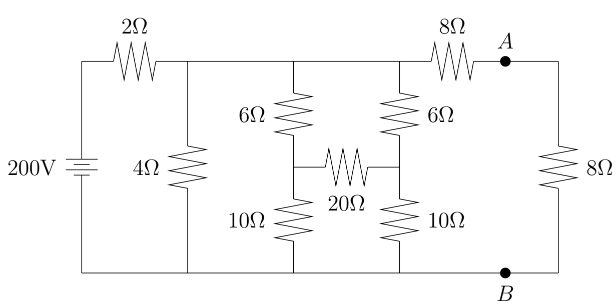
$$[(6 + 4) + 10 = 20]$$

(ii) ELECTRICAL AND ELECTRONICS ENGINEERING

E1. (a) Given the circuit shown below, find the condition under which the current through R will be zero.



(b) Find the current in 8Ω load across AB of the following circuit.



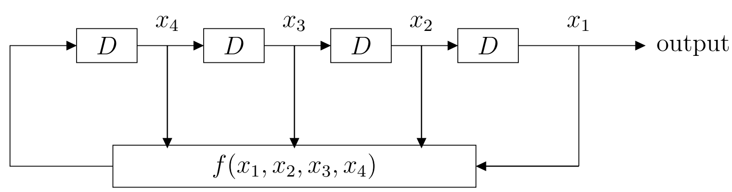
$[10 + 10 = 20]$

E2. (a) Consider two 2-bit unsigned integers $A = a_1a_0$ and $B = b_1b_0$. We like to compute $(A \cdot B) \bmod 3$.

(i) How many bits are required to represent the result?

(ii) Design a Boolean circuit using only two-input NAND gates that accepts $a_0,\ a_1,\ b_0$ and b_1 as inputs and computes the result.

(b) Consider the synchronous circuit consisting of four D flip-flops as shown in the figure below. All the flip-flops are driven by the same clock signal (not explicitly shown in the figure). This circuit generates a periodic binary output sequence '1100101000' repeatedly with the leftmost bit appearing first at the output. Initially, $x_1 = 1,\ x_2 = 1,\ x_3 = 0$ and $x_4 = 0$. Find a minimal expression for the combinational circuit $f(x_1, x_2, x_3, x_4)$.



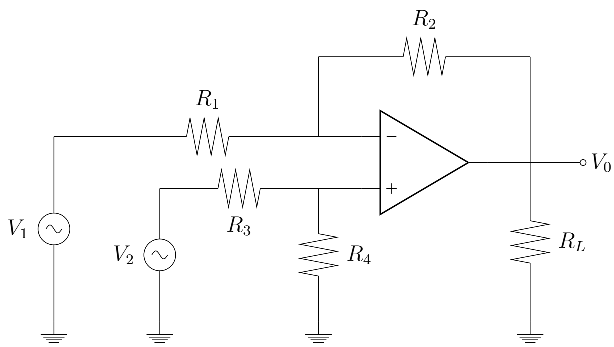
$[(2 + 8) + 10 = 20]$

E3. (a) The equivalent noise resistance for a multi-stage amplifier is the input resistance that will produce the same random noise at the output of the amplifier as the actual amplifier does.

The first stage of a two-stage amplifier has a voltage gain of 10, a 600Ω input resistance, a 1600Ω equivalent noise resistance and a 30 kΩ output resistance. For the second stage, these values are 25, 80 kΩ, 10 kΩ and 1 MΩ, respectively. Calculate the equivalent input noise resistance of this two-stage amplifier.

For an input noise resistance R, the noise voltage E_n generated at the input of a circuit is $E_n = \sqrt{4kT\Delta fR}$, where k is the Boltzmann constant, T is the temperature and Δf is the bandwidth.

(b) Consider the circuit shown below.



(i) What would be the value of V_0 for an ideal OP-AMP if $R_1 = R_2 = R_3 = R_4$?

(ii) What type of amplifier is represented by the above circuit?

$[10 + (8 + 2) = 20]$

E4. (a) Consider that only the four alphabets a, b, c, d are used in a communication between two parties. Studying a good amount of data, it has been noted that the expected probabilities of occurrences of these four alphabets are 0.41, 0.31, 0.21 and 0.07, respectively. You are required to encode these alphabets as binary strings so that the average length of the encoded bitstream is minimum. The lengths of the codewords may be different. The encoded binary string will not have any separating character between any two codewords and the recipient should be able to decode the binary string to revert back to the alphabets sent.

(i) Write down the four binary codewords corresponding to a, b, c and d.

(ii) Explain how you arrive at them.

(iii) What is the average bit-length per alphabet for your coding scheme?

(b) A frequency modulated (FM) signal is represented as

$$e = 12\sin(1.8\pi \times 10^8 t + 5\sin 400\pi t).$$

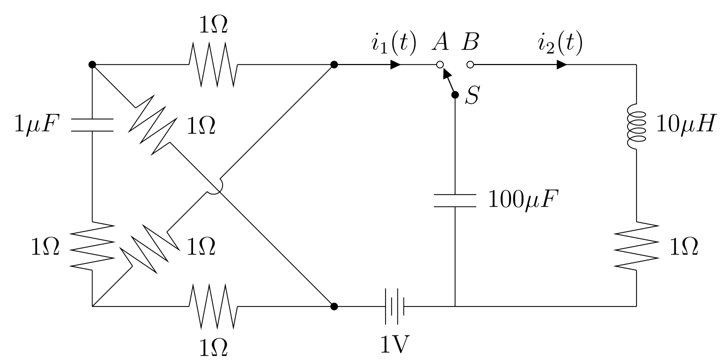
Find the carrier and modulating frequencies, and the maximum deviation of the FM wave. What power will this FM wave dissipate in a 10Ω resistor?

$[(2 + 6 + 2) + (2 + 2 + 3 + 3) = 20]$

E5. (a) Consider the following circuit where the capacitors and the inductor are ideal. At time $t = 0$, the switch S is open and at time $t = 1$, S is connected to node A. At time $t = 2$, S is disconnected from node A and connected to node B.

(i) Write an expression of the current $i_1(t)$.

(ii) Plot the waveform of $i_2(t)$ against time t.

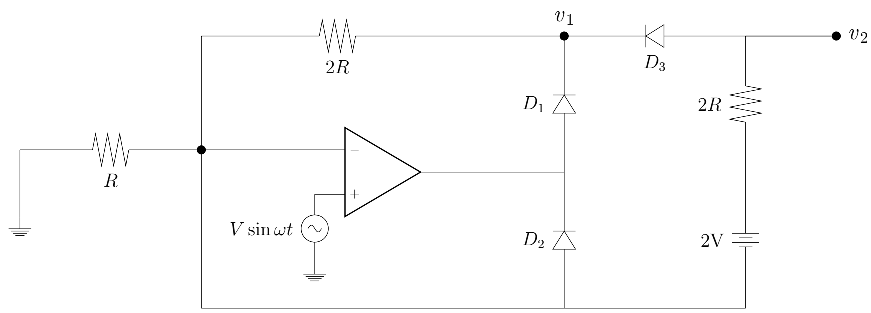


(b) Consider the following circuit with three resistors, three diodes D_1, D_2, D_3 and one AC source $V \sin wt$. Assume that the OP-AMP and the diodes are ideal.

(i) Show the voltage waveforms at v_1 and v_2 with respect to time t.

(ii) What is the maximum value of v_1 and v_2?

$[(6 + 4) + (6 + 4) = 20]$



E6. (a) A 50 Hz, 4-pole, 3-phase induction motor is running at 1200 rpm. It is connected to a 440V power line. It requires power input of 33 kW at 0.8 power factor lag. The motor has stator loss 1 kW and windage and friction loss 1.6 kW.

(i) Find the rotor copper loss.

(ii) Find the efficiency of the induction motor.

(b) A shunt generator delivers 100 kW at 250V and 400 rpm. The armature resistance is 0.02Ω and field resistance is 50Ω. The total contact drop for the brush is 2V. Calculate the speed of the machine running as a shunt motor and taking 50 kW input at 250V.

$[(5 + 5) + 10 = 20]$

E7. (a) A single phase 440/220V transformer has an effective primary resistance of 1Ω and secondary resistance of 0.11Ω. Its iron loss on normal input potential is 64W. Calculate maximum efficiency of the transformer at unity power factor.

(b) Let $f(x)$, $x = 0, 1, 2, \ldots, N$-1, be real periodic input data with periodicity N. Let $F(u), u = 0, 1, 2, \ldots, N$-1, be the discrete Fourier transform of $f(x)$. Compute the Fourier coefficients and phase angles for $f(x) = [10, 5, 7, 11, 17, 11, 7, 5]$ and $N = 8$, and indicate if these coefficients reflect any special property.

$[10 + (7 + 3) = 20]$

(iii) MATHEMATICS

M1. (a) Show that for any integer $p > 3$, the integers p, $p+2$ and $p+4$ cannot be simultaneously prime.

(b) Show that any perfect square is congruent to 0 or 1 modulo 4. Hence, or otherwise, prove that no integer in the sequence $11, 111, 1111, 11111, \ldots$ is a perfect square.

(c) For $n \geq 1$, let $\phi(n)$ be the number of positive integers less than or equal to n which are relatively prime to n. For example, $\phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \phi(4) = 2$.

(i) Find all n such that $\phi(n) = \frac{n}{2}$.

(ii) Find all n such that $\phi(n) = 4$.

$[5 + 6 + (4 + 5) = 20]$

M2. Let S_n be the group of all permutations of $\{1, \ldots, n\}$ under composition.

(a) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ be an element of S_6. Find the order of the cyclic subgroup generated by σ.

(b) Find the minimum n such that S_n contains a cyclic subgroup of order 30. Justify your answer.

(c) (i) Let S be a cyclic group of order 6. Show that S has a unique subgroup of order 3.

(ii) Let S be a finite cyclic group and K be a subgroup of S of order m. Show that an element $a \in S$ is an element of K if and only if $a^m = e$.

$[5 + 6 + (4 + 5) = 20]$

M3. Let $G = (V, E)$ be a graph and $e \in E$ be an edge whose endpoints are v_1 and v_2. Define the graph $G_e = (V', E')$ formed by merging v_1 and v_2 into a single vertex v as follows:

$$\begin{aligned} V' &= (V \setminus \{v_1, v_2\}) \cup \{v\}, \\ E' &= \Big\{E \setminus (\{e\} \cup \{(u, v_1) \in E\} \cup \{(u, v_2) \in E\})\Big\} \\ &\cup \Big\{(u, v) : (u, v_1) \in E \text{ or } (u, v_2) \in E\Big\}. \end{aligned}$$

Note, if there is a vertex $u \in V$ such that both (u, v_1) and (u, v_2) are in E, then the edge (u, v) occurs twice in G_e.

(a) If G is Eulerian, then is G_e Eulerian for all edges e of G? If G is not Eulerian, then is it possible for G_e to be Eulerian for some edge e of G? Justify your answer.

(b) Give an example of a bipartite graph G on at least 8 vertices and an edge e of G such that G_e is also bipartite.

(c) Give an example of a graph G on at least 6 vertices and an edge e in G such that the chromatic number of G_e is one less than the chromatic number of G.

[$7 + 7 + 6 = 20$]

M4. For variables x_1 and x_2, and $\alpha = (a_1, a_2)$, let the monomial $x_1^{a_1} x_2^{a_2}$ be denoted by x^α. Let A be a non-empty subset of pairs $\alpha = (a_1, a_2)$, where a_1 and a_2 are non-negative integers. For a field $\mathbb{F}$, define $I(A) = \{h_1 x^{\alpha_1} + \cdots + h_k x^{\alpha_k} : k \geq 0, \alpha_1, \ldots, \alpha_k \in A, h_1, \ldots, h_k \in \mathbb{F}[x_1, x_2]\}$.

(a) Show that $I(A)$ is an ideal of $\mathbb{F}[x_1, x_2]$.

(b) Suppose $g(x_1, x_2) = \displaystyle\sum_{i=1}^{\ell} g_i x^{\beta_i}$ with $g_i \in \mathbb{F}$ is in $I(A)$. Show that for $1 \leq i \leq \ell$, each x^{β_i} is divisible by some x^{α_i} with $\alpha_i \in A$.

(c) Let $A = \{(1, i), (j, 1) : i, j \geq 2\}$. Find a set of pairs S of minimum possible cardinality such that $I(A) = I(S)$.

(d) If I is an ideal of $\mathbb{F}[x_1, x_2]$, define

$$\sqrt{I} = \{f : f^m \in I \text{ for some } m \geq 0\}.$$

Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.

[$5 + 5 + 5 + 5 = 20$]

M5. (a) Show that the following function of two $n \times n$ matrices A and B with real entries defines an inner product of matrices.

$$\langle A, B\rangle = \mathrm{trace}\,(\mathrm{A^T B}),$$

where A^T is the transpose of A.

(b) The norm of a matrix A is defined by

$$\| A \| = \langle A, A\rangle^{\frac{1}{2}}.$$

(i) Show that $\| AB \| \leq \| A \| \, \| B \|$.

(ii) For a sequence of matrices $\{A_k\}$ and another matrix A, we define A_k converges to A if $\| A_k - A \| \to 0$ as $k \to \infty$. Show that if for a matrix A, $\{A^k\}$ converges to B, then $B^2 = B$.

(c) Let A and B be two $n \times n$ symmetric matrices such that $AB = BA$. Show that if $x \neq 0$ is an eigenvector of A and $Bx \neq 0$, then Bx is also an eigenvector of A corresponding to the same eigenvalue.

[$4 + (4 + 6) + 6 = 20$]

M6. (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n = 1, 2, \ldots$ with $| f_n'(x) | \leq 1$ for all n and x. Assume

$$\lim_{n \to \infty} f_n(x) = g(x),$$

for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

(b) Let $f : [0,1] \to \mathbb{R}$ be continuous over $[0,1]$, differentiable over $(0,1)$, $f(0) = 0$, and $0 \leq f'(x) \leq 2f(x)$.

Show that f is identically 0.

(c) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} (n+1) \int_0^1 x^n f(x) dx = f(1).$$

[6 + 6 + 8 = 20]

M7. (a) Let $\{a_n\}_{n\geq 0}$ be a non-negative sequence satisfying $a_{m+n} < a_m + a_n + 1$, for all positive integers m, n. Show that

(i) $a_{17} < 2a_8 + a_1 + 2$.

(ii) $\frac{a_n}{n}$ converges as $n \to \infty$.

(b) Let ε_n be the fractional part of $n!e$, where n is a positive integer.

(i) Show that $\frac{1}{n+1} < \varepsilon_n < \frac{1}{n}$ for all positive integers n.

(ii) Prove that $n \sin(2n!e\pi)$ converges to 2π as $n \to \infty$.

[(3 + 7) + (5 + 5) = 20]

(iv) PHYSICS

P1. (a) A particle of mass m and charge e moves in a magnetic field. The magnetic field is produced by a current I flowing in an infinite straight wire. The wire lies along the z-axis. The vector potential A of the induced magnetic field is given by

$$\begin{aligned} A_r &= A_\theta = 0, \\ A_z &= -(\frac{\mu_0 I}{2\pi}) \ln r, \end{aligned}$$

μ_0 being the permeability of vacuum. r, θ, z are cylindrical coordinates.

(i) Find the Lagrangian of the particle.

(ii) Write down the cyclic coordinates and the corresponding conserved momenta.

(b) In an inertial frame, two events have the space-time coordinates $\{x_1, y, z, t_1\}$ and $\{x_2, y, z, t_2\}$ respectively. Let $(x_2 - x_1) = 3c(t_2 - t_1)$, c representing the velocity of light in vacuum. Consider another inertial frame which moves along x-axis with velocity u with respect to the first one. Find the value of u for which the events are simultaneous in the later frame.

$[(6 + 2 + 2) + 10 = 20]$

P2. (a) (i) From the first and the second law of thermodynamics prove that for any system

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

The symbols have their usual meanings.

(ii) It is found experimentally that the product of pressure p and the volume V of a given gas is a function of temperature only. Also the internal energy of the gas depends only on temperature. Using (i), obtain the equation of the state of the gas.

(b) Two particles with masses m_1, m_2 and velocities $\vec{v_1}$, $\vec{v_2}$, respectively, stick together after collision. Find the velocity of the composite particle and also the loss in kinetic energy due to the collision.

$[(7+7)+6=20]$

P3. (a) The nozzle of a bicycle pump is blocked. With no force on the handle, the pump contains a volume V of air at 27^oC and 1 atmospheric pressure. The handle is now pushed down with a constant force F so as to reduce the volume to half. If no air escapes from the pump in the process and the change be adiabatic, compute the final temperature of air in the pump. Assume air to be an ideal gas. γ of air is 1.4 where γ is the ratio of specific heats at constant pressure and volume respectively. Given $2^{1.4} = 2.64$.

(b) Consider a system of N non-interacting quantum oscillators in equilibrium. The energy levels of a single oscillator are $E_n = (n + 1/2)\gamma$, where γ is a constant, $n = 0, 1, 2, \ldots$.

(i) Find the internal energy U and specific heat at constant volume C_v as functions of temperature T.

(ii) Draw a rough sketch of the variation of U and C_v with T.

(iii) Determine the equation of state for the system.

(iv) Calculate the fraction of particles at the n^{th} level.

$[6+(6+2+3+3)=20]$

P4. (a) Write down the Maxwell's equations in SI unit assuming that no dielectric or magnetic material is present. Hence justify what happens to the electric and magnetic fields in the following cases:

(i) If the signs of all the source terms are reversed,

(ii) if the system is space inverted, and

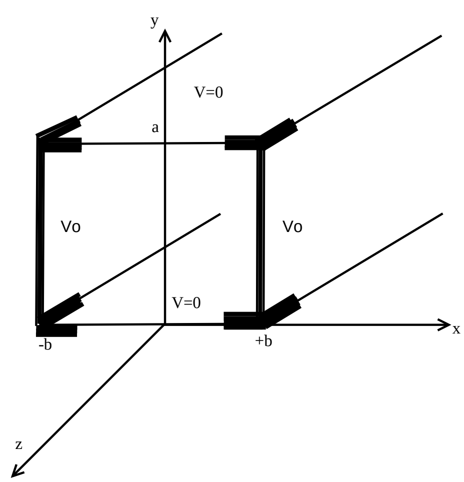
(iii) if the system is time reversed.

(b) (i) A particle moves in a time-independent electric field $\overrightarrow{E} = -\nabla\phi$ and an arbitrary magnetic field $\overrightarrow{B}$, ϕ being the scalar potential. Using Lorentz force law, prove that the energy of the particle is constant.

(ii) If the particle moves along x-axis in the electric field $\overrightarrow{E} = A \exp^{-t/\tau} \hat{\imath}$ (where A and τ are constants) and the magnetic field is zero along x-axis, find $x(t)$ with the initial conditions $x(0) = \dot{x}(0) = 0$.

$[2 + (2 + 2 + 2) + (6 + 6) = 20]$

P5. Two infinitely long grounded plates lie parallel to the x-z plane, one at $y = 0$ and the other at $y = a$. They are connected by metal strips maintained at constant potential V_0 as shown in the figure below. A thin layer of insulation at each corner of the metal strips prevents them from shorting out.



(a) Write Laplace's equation to calculate the potential inside the rectangular configuration thus formed.

(b) State the boundary conditions.

(c) Calculate the expression for potential inside the rectangular configuration.

(d) Draw a rough 3D sketch of this expression.

$[2 + 6 + 10 + 2 = 20]$

P6. (a) The mean momentum of a particle with wave function $\psi(x)$ is $< p >$. Show that the mean momentum is $< p > +p_o$ for the wave function $e^{ip_0\hat{x}/h}\psi(x)$, where $\hat{x}$ is the position operator and p_0 is a constant.

(b) A particle is in the ground state of a one dimensional box of length L. Suddenly the box expands to twice its size, leaving the wave function undisturbed. Find the probability of finding the particle in the ground state under measurement of Hamiltonian observable.

(c) $\hat{x}$, $\hat{p}$ and $\hat{H}$ are respectively, position, momentum and Hamiltonian operator for harmonic oscillator. Show that $[\hat{x}, \hat{H}] = \frac{i\hbar\hat{p}}{m}$, where m is the mass of the harmonic oscillator.

$[5 + 10 + 5 = 20]$

P7. (a) An electric boiler consists of 2 heating elements, each of 220V, 5 kW rating, connected in series. It contains 10 litres of water at 20^oC. On an average, there is a 20% loss of heat for this boiler. Calculate the time required for the water to boil at normal atmospheric pressure. Calculate the same if the heating elements are connected in parallel. What conclusion can you draw from the ratio of the above two results? [Given J = 4200 Joules/kcal, and the mass of 1 litre of water is 1 kg.]

(b) In a centre-tap full wave rectifier, the load resistance is 5 kΩ. The AC supply across the primary winding is $220 \sin \omega t$.

If the transformer turn ratio is 1:2, compute the DC load voltage (V_{DC}) and current (I_{DC}). Assume negligible winding resistance. [Hint: For half wave rectifier, $V_{DC} = V_m \ / \ \pi$.]

$[(6 + 6 + 2) + 6 = 20]$

(v) STATISTICS

S1. (a) Let $X_1, X_2, \ldots, X_n$ (n ≥ 3) be identically distributed random variables with $Var(X_1) = 1$ and $Cov(X_i, X_j) = \rho < 0$ for all $i \neq j$. For any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in R^n$ (the superscript T stands for transpose), define $S(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \alpha_i X_i / \|\boldsymbol{\alpha}\|$.

(i) Find $\boldsymbol{\alpha}$ such that $V(\boldsymbol{\alpha}) = Var(S(\boldsymbol{\alpha}))$ is maximum.

(ii) If it is known that $\|\boldsymbol{\alpha}\| = 1$ and $\alpha_1 > 0$, is this maximizer of $V(\boldsymbol{\alpha})$ unique? Justify your answer.

(iii) Show that $V(\boldsymbol{\alpha})$ cannot exceed $n/(n-1)$.

(b) Let $X_1, X_2, \ldots, X_n$ be independent random variables, where $X_i \sim N(\mu_i, \sigma^2)$ for all $i = 1, 2, \ldots, n$. Show that for any $C > 0$, $P(\sum_{i=1}^{n}(X_i - \mu_i)^2 \leq C\sigma^2) \to 0$ as $n \to \infty$.

[(6 + 2 + 6) + 6 = 20]

S2. Consider a linear model $Y_i = \mathbf{x}_i^T \boldsymbol{\beta} + \epsilon_i$, $i = 1, 2, \ldots, n$, where $\mathbf{x}_i = (1, x_{i1}, x_{i2}, \ldots, x_{ip})^T$, $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \ldots, \beta_p)^T$, and the superscript T stands for transpose. Assume that the $\mathbf{x}_i$'s are non-stochastic and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent, where $\epsilon_i \sim N(0, \sigma^2 i)$ for all i.

(a) Consider p non-zero real numbers $l_1, l_2, \ldots, l_p$. Derive the condition under which $l_1\beta_1 + l_2\beta_2 + \ldots + l_p\beta_p$ is estimable.

(b) Assuming β_1 is estimable, find the best linear unbiased estimator for β_1.

(c) Is the estimator in (b) the uniformly minimum variance unbiased estimator? Justify your answer.

(d) Check whether the estimator in (b) is consistent.

[5 + 5 + 5 + 5 = 20]

S3. (a) If X_1 and X_2 jointly follow a bivariate normal distribution with $E(X_i) = \mu_i$, $Var(X_i) = \sigma_i^2$, $i = 1, 2$ and $Corr(X_1, X_2) = \rho$, find the probability $P(X_1 > \mu_1, X_2 < \mu_2)$.

(b) Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$. Assume that there exist positive constants C_0, C_1 and C_2 such that $|\mu_i| \leq C_0$ and $C_1 \leq \sigma_i^2 \leq C_2$ for $i = 1, 2, \ldots, n$. For any fixed $x \in R$, show that $\min\{|X_1 - x|, |X_2 - x|, \ldots, |X_n - x|\} \xrightarrow{P} 0$ as $n \to \infty$.

[$10 + 10 = 20$]

S4. (a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed $N(\mu, 1)$ variables. For any fixed $t \in R$, find the uniformly minimum variance unbiased estimators for $\Phi(t - \mu)$ and $\phi(t - \mu)$, where Φ and ϕ respectively denote the cumulative distribution function and the density function of the standard normal variable.

(b) Let A_θ be the square with vertices as (θ,0), (0,θ), (-θ,0) and (0,-θ). Let f_θ be the uniform distribution over A_θ. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n independent random samples from f_θ. Find the maximum likelihood estimate for θ, assuming θ is a positive real number.

[$(6 + 4) + 10 = 20$]

S5. (a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed $U(0, 1)$ random variables. Find the distribution of $D_n = \max_{1 \leq i < j \leq n} |X_i - X_j|$. Also show that $D_n \xrightarrow{P} 1$ as $n \to \infty$.

(b) Consider a two-class classification problem, where the two competing classes have equal prior probability 1/2 and their density functions are given by $f_1(x) = 1$ and $f_2(x) = 1 + \sin(2k\pi x)$, where $0 \leq x \leq 1$ and k is a positive integer.

(i) Find the classification rule that has the minimum misclassification probability.

(ii) Compute the misclassification probability and show how it varies with k.

$[10 + (2 + 6 + 2) = 20]$

S6. (a) Let A and B be two boxes containing m and n balls (m and n are even integers), respectively. Consider the following sequence of trials. In each trial, a box is selected at random with equal probability; one ball is taken from the selected box and it is placed in the other box. The sequence of trials comes to an end when one of the boxes becomes empty. Find the probability that box A becomes empty.

(b) A d-dimensional random vector $\mathbf{X}$ is called spherically symmetric about $\boldsymbol{\theta} \in R^d$ if $(\mathbf{X} - \boldsymbol{\theta})$ and $H(\mathbf{X} - \boldsymbol{\theta})$ have the same distribution for all orthogonal matrices H.

Show that $\mathbf{X}$ is spherically symmetric about $\boldsymbol{\theta}$ if and only if its characteristic function $\psi(\mathbf{t}) = E(e^{i\mathbf{t}^T\mathbf{X}})$ is of the form $\psi(\mathbf{t}) = e^{i\mathbf{t}^T\boldsymbol{\theta}} g(\|\mathbf{t}\|)$ for g being a function defined on $[0, \infty)$. The superscript T stands for transpose.

$[10 + (4 + 6) = 20]$

S7. (a) Let $X_1, X_2, \ldots, X_n$ be a sample from P_θ, $\theta \in \Theta$, and T be a real-valued complete and sufficient statistic for θ. Let S be another real-valued sufficient statistic. Show that $T = g(S)$ for a suitable measurable function $g : R \to R$.

(b) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed $N(\mu, 1)$ variables. Find the uniformly most powerful test of level α for $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$. Also show that it is the least powerful test among all size α tests for $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$.

$[10 + (4 + 6) = 20]$