2016

Booklet No. TEST CODE: REI

Forenoon

Questions: 10	**Time: 2 hours**

- *On the answer booklet write your Name, Registration number, Test Code, Number of the booklet etc. in the appropriate places.*
- This test has 10 questions. Answer *ALL* ten. All questions carry equal marks (10).

1. Which one is greater: $99^{50} + 100^{50}$ *or* 101^{50}?
   [10]

2. Suppose a, b, c are real numbers such that $a+b+c = 3$, $a^2+b^2+c^2 = 5$ and $a^3+b^3+c^3 = 4$. Then, find the value of abc.
   [10]

3. If A is the real $n \times n$ matrix $((a_{ij}))$ with $a_{ij} = \rho^{|i-j|}$, then show that $|A| = (1-\rho^2)^{n-1}$.
   [10]

4. Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, and continuously differentiable on (a,b). Prove that if $f(a) = f(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.
   [10]

5. Let $U : [0,1] \to \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ be two functions. Suppose U is differentiable at all $x \in [0,1]$ and for every x, $y \in [0,1]$, the following holds:

$$U(x) \geq U(y) + (x-y)f(y).$$

   Show the following:
   (a) Derivative of U at x is $f(x)$ for all $x \in [0,1]$.
   (b) U is convex.
   (c) f is non-decreasing.
   [4+3+3=10]

6. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}_+$, defined as follows: $f(x) = \max(x, x^2)$ for all $x \in \mathbb{R}_+$.
   (a) Plot f.
   (b) Find a point where f is not differentiable and argue why it is not differentiable.
   [5+5=10]

7. Solve the following optimization problem. Maximize

$$3xy - y^2$$

subject to

$$2x + 5y \geq 20,$$
$$x - 2y = 5,$$
$$x, y > 0.$$

Show the solution graphically.
[10]

8. Let X, Y be random variables such that $(X, Y) \in \{(x, y) | 0 \leq x \leq 3, 0 \leq y \leq 4\}$ always. The joint cumulative distribution function (cdf) of X and Y in this rectangle is $F(x, y) = \frac{xy(x^2+y)}{156}$.
Find
(a) $P(1 \leq X \leq 2 \; and \; 1 \leq Y \leq 2)$.
(b) The cdf of Y.
(c) The joint probability density function (pdf) of X and Y.
[4+3+3=10]

9. An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly one ace.
[10]

10. Consider an experiment to toss two balls into four boxes in a way such that each ball is equally likely to fall in any box. Let X denote the number of balls in the first box.
(a) What is the cumulative distribution function (cdf) of X?
(b) What is the probability mass function (pmf) of X?
(c) Find the mean and variance of X.
[3+3+4=10]