Booklet No. TEST CODE: QEA

Forenoon

Time: 2 hours

- Write your Registration Number, Test Code, Question Booklet Number etc. in the appropriate places of the Answer Booklet.

- This test has **10** questions. **Answer as many as you can**. All questions carry equal (10) marks.

1. (a) If $|a| < 1$ and $|b| < 1$, then find if the series

$$a(a+b) + a^2(a^2 + b^2) + a^3(a^3 + b^3) + \ldots\ldots$$

converges or not.

(b) Let x_1 and x_2 be the roots of the quadratic equation $x^2 - 3x + a = 0$ and x_3 and x_4 be the roots of the quadratic equation $x^2 - 12x + b = 0$. If x_1, x_2, x_3 and x_4 $(0 < x_1 < x_2 < x_3 < x_4)$ are in G.P., then find the value of ab.

2. A bivariate probability density function is defined by

$$f_{X,Y}(x, y) = C(x + 2y) \text{ if } 0 < y < 1 \text{ and } 0 < x < 2$$
$$= 0 \text{ otherwise}$$

where C is a constant.

(a) Find the value of C.

(b) Find the marginal distribution of X.

(c) Find the joint cumulative distribution function of X and Y.

3. For the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, show that there does not exist any invertible matrix Q and a diagonal matrix D such that $Q^{-1} A\, Q = D$.

4. Let $f(x)$ = minimum $\{x, 10 - x\}$, $x \geq 0$. For any non-negative real number t, let $x(t)$ be the (global) maxima of $f(x)$ for $x \in [0, t]$. Find the function $x(t)$.

5. Find all positive solutions of the following system of equations:

$$x_1 + x_2 = x_3^2,$$
$$x_2 + x_3 = x_4^2,$$
$$x_3 + x_4 = x_5^2,$$
$$x_4 + x_5 = x_1^2,$$

and $$x_5 + x_1 = x_2^2.$$

6. Suppose a real-valued function f over $[0,\infty)$, satisfies the following properties: (a) $f(x)$ is continuous for $x \geq 0$, (b) $f'(x)$ exists for $x > 0$, (c) $f(0) = 0$, and (d) $f'(x)$ is monotonically increasing.
Now define another real-valued function g over $(0,\infty)$, as $g(x) = \dfrac{f(x)}{x}$ for $x > 0$.
Show that $g(x)$ is a monotonically increasing function.

7. Let f be a real valued function defined on 2×2 real matrices $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ where $a_1 = (a_{11}, a_{12})'$ and $a_2 = (a_{21}, a_{22})'$ are any two real 2-dimensional row vectors. Further, f satisfies the following properties.

(i) f is a linear function of each row when the other row is held fixed. For example, when the second row is held fixed,

$$f\begin{pmatrix} \delta\, a_1 + a_1^* \\ a_2 \end{pmatrix} = \delta\, f\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + f\begin{pmatrix} a_1^* \\ a_2 \end{pmatrix}$$

where a_1^* is any real 2-dimensional row vector and δ is any real number.

(ii) $f\begin{pmatrix} a_1 \\ a_1 \end{pmatrix} = 0$ for all a_1.

(iii) $f\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 1$ when $a_1 = (1,0)$ and $a_2 = (0,1)$

Show that f is the determinant of A.

8. A deck of 52 playing cards is shuffled, and the cards are turned up one at a time until the first 'ace' appears. Find which of the following two events A and B is more likely to happen.
A: The next card drawn (i.e., the card following the first 'ace') is the 'ace' of spades; and
B: The next card drawn is the '2' of clubs.

9. Let $f:\Re \to \Re$ be a twice differentiable function such that $f(\xi_i)=0$ for $i=1,2,3$ where $\xi_i's$ are distinct. Show that the second derivative of f vanishes at a point.

10. Find the value of the integral

$$\int_A x^2 e^{xy} dxdy$$

where A is the region bounded by the straight lines y = x, y = 0, and x = 1 .