2014

Booklet No.

TEST CODE: QEA

Forenoon

Questions: 10	Time: 2 hours

- Write your Name, Registration Number, Test Code, Question Booklet Number etc. in the appropriate places of the answer booklet.

- This test has **10** questions. **Answer as many as you can**. All questions carry equal (10) marks.

1. The derivative of x^2 with respect to x is $2x$. However, let us write x^2 as the sum of x x's, and then take the derivative. That is, let $f(x) = x + x + ... + x$ (x times), then

$$f'(x) = \frac{d}{dx}[x] + \frac{d}{dx}[x] + ... + \frac{d}{dx}[x] \text{ (x times)} = 1 + 1 + ... + 1 \text{ (x times)} = x .$$

This argument appears to show that the derivative of x^2 with respect to x is actually x. Where is the fallacy? Explain clearly.

2. Find the number of nonnegative integer solutions of the equation $x_1 + x_2 + x_3 = 10$.

3. Suppose A is an $n \times n$ symmetric *positive definite* matrix and B is an $n \times m$ matrix with *rank* (B) = m. Prove that $B^T AB$ is also a symmetric *positive definite* matrix. (B^T denotes the transpose of matrix B.)

4. Let $f : [1,4] \to \Re$ be a continuous function and $f(1) = f(4)$. Prove that there exists at least one point $\xi \in [1,4]$ such that $f(\xi) = f(\xi + 1.5)$.

5. Suppose f_n, $n = 1,2,3,...$, is a sequence of real valued functions defined on a set $S \subseteq \Re$. We say that the sequence f_n converges pointwise to a function f defined on S if $\lim_{n\to\infty} f_n(x) = f(x), \ \forall x \in S$.

   Let $f_n(x) = \left(1 - |x|\right)^n$, $\forall x \in (-1,1)$. Find the function f to which this sequence converges pointwise.

6. Consider the following definitions.

   Convex Combination: A vector $y \in \Re^n$ is said to be a convex combination of the vectors $x^1, x^2, ..., x^m \in \Re^n$ if there exist m non-negative real numbers $\theta_1, \theta_2, ..., \theta_m$ such that (i) $\sum_{i=1}^{m} \theta_i = 1$, and (ii) $y = \sum_{i=1}^{m} \theta_i x^i$.

   Convex Set: A set $S \subset \Re^n$ is a *convex set* if for every two vectors in S, all convex combinations of these two vectors are also in S.

   Use the method of induction to prove that if a set $S \subset \Re^n$ is convex, then for any integer $m > 1$ and for any m vectors in S, every convex combination of these m vectors is in S.

7. Let X, Y be random variables such that $(X,Y) \in \{(x,y) | 0 \le x \le 3,\ 0 \le y \le 4\}$ always. The joint cumulative density function (c.d.f) of X and Y in this rectangle is

$$F(x,y) = \frac{xy(x^2+y)}{156}.$$

Find

(a) $P(1 \le X \le 2 \text{ and } 1 \le Y \le 2)$,

(b) the cumulative density function of Y,

(c) the joint probability density function of X and Y,

(d) $P(Y \le X)$.

8. (a) Suppose a surveillance system has a 99% chance of correctly identifying a terrorist and a 99.9% chance of correctly identifying someone who is not a terrorist. Suppose there are 1000 terrorists in an adult male population of 300 million, and one of these 300 million men is randomly selected, scrutinised by the system, and identified as a terrorist. Then the probability that he is actually a terrorist is closest to

(i) 0.99, (ii) 2/3, (iii) 1/3, (iv) 1/300.

Justify your answer.

(b) A class contains 10 boys and 15 girls. 8 students are to be selected at random from the class without replacement. Let X be the number of boys selected and Y the number of girls selected. Find $E(X-Y)$. Be sure to give the reason for each step in your answer.

9. Let $A \subset \Re^2$ be open and $f : A \to \Re$ be twice continuously differentiable. Consider the problem of maximizing $f(x,y;a)$ with respect to $(x,y) \in \Re^2$, where $a \in \Re$ is a *parameter* for the maximization problem.

Given any $a \in \Re$, suppose $\left(x^*(a), y^*(a)\right)$ is a solution to the maximization problem. Provide (with a clear explanation) a *sufficient* condition for $x^*(a)$ and $y^*(a)$ to be *continuously differentiable* functions of a.

10. Recall the following version of Lagrange Theorem for optimization. *Let $A \subset \Re^2$ be open, and $f : A \to \Re$ and $g : A \to \Re$ be continuously differentiable functions on A. Suppose $\left(x^*, y^*\right)$ is a point of local maximum or minimum of $f(x,y)$ subject to the constraint $g(x,y) = 0$. Suppose further that $\left(\frac{\partial g}{\partial x}(x^*, y^*), \frac{\partial g}{\partial y}(x^*, y^*)\right) \neq (0,0)$. Then there exists $\lambda^* \in \Re$ such that $\dfrac{\partial f}{\partial x}(x^*, y^*) = \lambda^* \dfrac{\partial g}{\partial x}(x^*, y^*)$, and $\dfrac{\partial f}{\partial y}(x^*, y^*) = \lambda^* \dfrac{\partial g}{\partial y}(x^*, y^*)$.*

Let *f* and *g* be functions on $\Re^2$ defined respectively by

$$f(x,y) = \tfrac{1}{3}x^3 - \tfrac{3}{2}y^2 + 2x ,$$

and

$$g(x,y) = x - y .$$

Consider the problems of *maximizing* and *minimizing* *f* on the set $C = \left\{(x,y) \middle| g(x,y) = 0\right\}$.

Examine whether you can apply the Lagrange Theorem to solve these *maximization* and *minimization* problems. If yes, find the solutions. If not, provide clear explanations.