JRF Mathematics Examination RM II

Solve any six questions.

- 1. Let $n \geq 3$ be a natural number. Prove that the three cycle (1,2,3) is not a cube of any element in the symmetric group S_n .
- 2. Prove that any group of order 35 is cyclic.
- 3. (a) Let F be a field of odd characteristic and let K be a field extension of F of degree 2. Prove that there exists an $a \in F$ such that $K \cong F(\sqrt{a})$.
 - (b) Give an example of a degree 2 extension K of a field F of characteristic two which is not obtained by attaching a square root of an element of F.
- 4. Let $r \leq n$ be natural numbers and let $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ be two linearly independent subsets of \mathbb{R}^n such that

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle \ \forall \ 1 \le i, j \le r,$$

where \langle , \rangle denotes the standard inner product on \mathbb{R}^n . Prove that there exists an orthogonal operator T on \mathbb{R}^n such that $T(v_i) = w_i$ for all $1 \leq i \leq r$.

5. Show that the space C[0,1] of real-valued continuous functions on the unit interval [0,1] with the sup norm

$$||f|| = \sup_{x \in [0,1]} |f(x)|$$

is not a Hilbert space with respect to any inner product.

- 6. Let \mathcal{H} be a Hilbert space with complete orthonormal basis $\{e_n|n\in\mathbb{N}\}$. Let R be the shift operator on \mathcal{H} defined by $R(e_n)=e_{n+1}$, and extended by linearity and continuity. Show that there is no $x\in\mathcal{H}$, such that $\mathrm{Span}\{R^{2n}(x)|n\geq 0\}$ is dense in \mathcal{H} .
- 7. Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$. Find the eigenvalues of A and identify the set

$$E = \{a \in \mathbb{R} : \lim_{n \to \infty} a^n A^n \text{ exists and is different from zero} \}.$$

Note that convergence of a sequence of matrices is taken entrywise.

- 8. Let T be a bounded operator on a normed linear space X such that $T^2 = T$. Compute the inverse of $\lambda I T$, for any complex number $\lambda \neq 0, 1$.
- 9. Prove that for any natural number n, there exist n consecutive integers each of which is divisible by a perfect square greater than one.
- 10. Let $\{a_1, \ldots, a_{n^2+1}\}$ be a permutation of the set $\{1, 2, \ldots, n^2+1\}$. Prove that the sequence $\{a_i\}$ contains a monotone subsequence of length n+1.
- 11. Let p > 3 be a prime number and \mathbb{F}_p denote the finite field of order p. Prove that the polynomial $X^2 + X + 1$ is reducible in $\mathbb{F}_p[X]$ if and only if $p \equiv 1 \pmod{3}$.