2015

BOOKLET No. TEST CODE : MTB

Afternoon

Answer as many questions as you can

Answering *five* questions correctly would be considered adequate

Time : 2 hours

Write your Registration number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer-booklet.

ALL ROUGH WORK IS TO BE DONE ON THIS BOOKLET
AND/OR THE ANSWER-BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP! WAIT FOR THE SIGNAL TO START.

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{Q}$ denotes the set of rational numbers.
- $\mathbb{Z}$ denotes the set of integers.

Q 1. Let f be a measurable function on $\mathbb{R}$ such that $\int_I f d\lambda = 0$ for all bounded intervals $I \subset \mathbb{R}$, where λ is the Lebesgue measure on $\mathbb{R}$.
Show that

$$\lambda(\{x \in \mathbb{R} : f(x) \neq 0\}) = 0.$$

Q 2. Let $C_{\mathbb{R}}[0,1]$ denote the Banach space of all continuous real-valued functions on $[0,1]$ with norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}.$$

Show that there is no inner product on $C_{\mathbb{R}}[0,1]$ that induces the norm $\|\cdot\|_\infty$.

Q 3. Find the smallest positive integer n such that

$$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

is isomorphic to a subgroup of S_n, where S_n is the group of all permutations of $\{1, 2, \ldots, n\}$.

Q 4. Let $p = 1 + 4n$ be a prime in $\mathbb{Z}$.

(a) Show that $(2n)!$ is a solution to the congruence

$$x^2 \equiv -1 \pmod{p}.$$

(b) Use (a) to show that p is not a prime element in $\mathbb{Z}[i]$.

(c) Use (b) to show that $p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$.

Q 5. Fix a $p \in [1, \infty]$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers such that $\{a_n x_n\}_{n=1}^\infty \in \ell^p$ for all sequences $\{x_n\}_{n=1}^\infty \in \ell^p$. Show that $\{a_n\}_{n=1}^\infty \in \ell^\infty$.

Q 6. Let $\{B_\alpha : \alpha \in A\}$ be a family of piecewise disjoint Lebesgue measurable subsets of $[0,1]$ each of positive measure. Show that the family is countable.

Q 7. Let $C[0,1]$ denote the Banach space of all continuous complex-valued functions on $[0,1]$ with norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}.$$

Let $g : [0,1] \to [0,1]$ be a non-constant continuous function. Consider the operator $T : C[0,1] \to C[0,1]$ defined by $Tf = f \circ g$, that is, $Tf(x) = f(g(x)), f \in C[0,1], x \in [0,1]$.

Show that the image of the unit ball in $C[0,1]$ under T is not compact in $C[0,1]$.

Q 8. Let p be a prime and $K = \mathbb{Q}(\alpha)$, where $\alpha^3 = p$. Find the minimal polynomial of $\alpha + \alpha^2$ over $\mathbb{Q}$. Justify your answer completely.

Q 9. Let $n > 1$ be an odd integer. Show that n does not divide $3^n + 1$.

Q 10. (a) Let R be a commutative ring with unity. Let $a, b, c \in R$ be such that there exist $x, y, z \in R$ with

$$xa + yb + zc = 1.$$

Show that there exist $\alpha, \beta, \gamma \in R$ such that

$$\alpha a^{15} + \beta b^{16} + \gamma c^{17} = 1.$$

(b) Let R be a commutative ring with unity such that for each $a \in R$ there exists an integer $n(a) > 1$ such that $a^{n(a)} = a$. Prove that every prime ideal of R is maximal.