2015

BOOKLET No. TEST CODE : MTA

Forenoon

Answer as many questions as you can

Answering *five* questions correctly would be considered adequate

Time : 2 hours

Write your Registration number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer-booklet.

ALL ROUGH WORK IS TO BE DONE ON THIS BOOKLET
AND/OR THE ANSWER-BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP! WAIT FOR THE SIGNAL TO START.

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.
- $\mathbb{N}$ denotes the set of positive integers.

Q 1. Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n}.$$

Q 2. Let A be an $m \times n$ real matrix.

(a) Show that $\mathrm{N}(A) \cap \mathrm{Im}(A^T) = \{0\}$, where A^T is the transpose of A, $\mathrm{Im}(A)$ is the image of A and $\mathrm{N}(A) = \{v \in \mathbb{R}^n : Av = 0\}$.

(b) If for two suitable matrices B and C, $AA^T B = AA^T C$ then show that $A^T B = A^T C$.

Q 3. Let V be a finite dimensional vector space over $\mathbb{R}$. Suppose that a subset $A \subset V$ has the following property: For any finite set of scalars $a_1, a_2, \ldots, a_n \in \mathbb{R}$ which satisfies $\sum_{i=1}^{n} a_i = 1$ and any vectors $v_1, v_2, \ldots, v_n \in A$, $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in A$.

Show that $A = x_0 + W$ for some $x_0 \in V$ and some subspace W of V, where $x_0 + W = \{x_0 + v : v \in W\}$.

Q 4. Let X_n = number of heads obtained from n independent coin tosses with probability of head p. Let p_n be the probability that X_n is an even number.

(a) Show that $p_{n+1} = (1 - 2p)p_n + p$.

(b) Show that $\lim_{n\to\infty} p_n$ exists and find the limit.

Q 5. Let C be a closed subset of $\mathbb{R}^n$ and r be a positive real number. Show that the set

$$D = \{y \in \mathbb{R}^n : \text{there exists } x \in C \text{ with } \|x - y\| = r\}$$

is closed in $\mathbb{R}^n$, where $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Q 6. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

(a) Compute $\int_0^{2\pi} f(re^{i\theta})\, d\theta$, $r > 0$.

(b) If

$$\int_{\mathbb{R}^2} |f(x+iy)|\, dxdy < \infty,$$

then prove that $f(z) = 0$ for all $z \in \mathbb{C}$.

Q 7. Let $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, $t \in \mathbb{R}$, be the standard normal density function and $\Phi(x) = \int_{-\infty}^{x} \phi(t)\, dt$ be the standard normal distribution function. Let

$$f_\alpha(t) = 2\phi(t)\Phi(\alpha t), \quad t \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$. Show that f_α is a probability density function for any $\alpha \in \mathbb{R}$.

Q 8. Consider the equivalence relation '$\sim$' on $\mathbb{R} \times [0,1]$ defined by

$$(x,t) \sim (x+1,t), x \in \mathbb{R} \text{ and } t \in [0,1].$$

Let $X = \big(\mathbb{R} \times [0,1]\big)/\sim$ be the quotient space. Prove that X is Hausdorff and compact.

Q 9. For $k \in \mathbb{N}$, let

$$P_k(t) = c_k\Big(\frac{1+\cos t}{2}\Big)^k, t \in [-\pi, \pi],$$

where $c_k \in \mathbb{R}$ is chosen in such a way that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_k(t)\, dt = 1.$$

(a) Prove that $c_k \leq \frac{\pi}{2}(k+1)$ for all $k \in \mathbb{N}$.
(b) Using (a) or otherwise prove that for every δ with $0 < \delta < \pi$,

$$\lim_{k\to\infty}\Big(\sup_{\delta \leq |t| \leq \pi} P_k(t)\Big) = 0.$$

Q 10. Let $\Omega = \{z \in \mathbb{C} : |z| < 2\}$ and f be a function on Ω which is holomorphic at every point of Ω except $z = 1$ and at $z = 1$ it has a simple pole. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1.$$

Prove that $\lim_{n\to\infty} a_n = -c$, where c is the residue of f at $z = 1$.