2014

BOOKLET No. TEST CODE : MTA

Forenoon

Answer as many questions as you can

Answering *five* questions correctly would be considered adequate

Time : 2 hours

Write your Registration number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer-booklet.

ALL ROUGH WORK IS TO BE DONE ON THIS BOOKLET
AND/OR THE ANSWER-BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP! WAIT FOR THE SIGNAL TO START.

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.

Q 1. Let (X, d) be a compact metric space. Let $f : X \to X$ be a function such that

$$\text{graph}(f) := \{(x, f(x)) : x \in X\},$$

is closed. Prove that f is continuous.

Q 2. Let X be a countable infinite set. Prove that there does not exist any topology on X such that X is connected, normal, and all singletons in X are closed.

Q 3. Let $\{f_n\}_{n\geq 1}$ be a sequence of continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ converging uniformly. Suppose

$$\lim_{n\to\infty} f_n(p, q) = 0,$$

for all rational numbers p and q. Prove that

$$\lim_{n\to\infty} f_n(x, y) = 0,$$

for all $(x, y) \in \mathbb{R}^2$.

Q 4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the partial derivatives f_x and f_y at $(0, 0)$.

(ii) Are f_x and f_y continuous at $(0, 0)$?

(iii) Is f differentiable at $(0, 0)$?

Justify your answers.

Q 5. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function. Let λ be the Lebesgue measure on $[-1, 1]$. Suppose $|\int_A f d\lambda| \leq \lambda(A)$ for all measurable sets $A \subseteq [-1, 1]$. Prove that the range of f is contained in $[-1, 1]$.

Q 6. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of all the rational numbers. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n).$$

Prove that f and g are integrable on $\mathbb{R}$.

Q 7. Let $f : [0, \infty) \to \mathbb{C}$ be a non-constant function. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = f(|z|).$$

Prove that g is not holomorphic.

Q 8. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that for every $z \in \mathbb{C}$, there is some integer $n \geq 0$ satisfying $f^{(n)}(z) = 0$. Prove that f is a polynomial.

[Here $f^{(n)}$ is the n-th derivative of f if $n \geq 1$, and $f^{(0)} = f$.]

Q 9. Find the general solution (with appropriate justification) to the differential equation given by

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 13x = 0,$$

for $t > 0$.

Q 10. Consider the Bernoulli differential equation given by

$$\frac{dy}{dt} = y(9 - y^2),$$

for $t > 0$. Prove the existence and uniqueness of the above equation, when

(i) $y(0) = 1$, and
(ii) $y(0) = 0$.