2016

BOOKLET NO. TEST CODE: **STB**

Afternoon

No. of Questions: 8 **Time: 2 hours**

- **Answer as many questions as you can. All questions carry equal weight.**
- **Do not feel discouraged if you are not able to answer all the questions.**
- **Partial credit may be given for partial answer.**
- **Full credit will be given for complete and rigorous arguments.**

Write your Name, Registration Number, Test Code, Booklet No. etc., in the appropriate places on the answer-booklet.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER-BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP! WAIT FOR THE SIGNAL TO START.

1. Consider a random arrangement of 20 boys and 16 girls in a line. Let X be the number of boys with girls on both sides, and Y be the number of girls with boys on both sides. Find $E(X+Y)$.

2. Let $X_1, ..., X_n$ be idependently and identically distributed random variables with distribution function F, where n is an odd integer. Let $Y_1, ..., Y_n$ be independently and identically distributed observations from the empirical distribution function $\widehat{F}_n$ associated with $X_1, ..., X_n$. Obtain the distribution of the median of $Y_1, ..., Y_n$.

3. Let X be a nonnegative random variable such that

$$E\left(\sum_{n=1}^{\infty} X^n\right) < \infty. \qquad (*)$$

   (a) Show that X cannot be uniformly distributed over $(0, 1)$.

   (b) Show that $P(X \geq 1) = 0$.

   (c) Give an example of a continuous random variable X satisfying the relation in (*).

4. A shopkeeper places an order of one extra item only when his stock of that item at the end of a week is one or less. Let $X_n,\ n \geq 1$, denote the number of items in his stock at the end of the n-th week before he decides to place an order or not, with $X_0 = 0$. Assume that the weekly demand for the sale of this item from this shop follows a Poisson distribution with mean λ, and the item is sold on demand till the stock lasts.

   Prove that $X_n,\ n \geq 1$, defines a Markov chain, and then specify its state space and the transition probability matrix. Also derive the stationary probabilities.

5. Let $(X_1, Y_1), ..., (X_n, Y_n)$ be independently and identically distributed observations from the uniform distribution over the triangle ABC, where $A = (0, 0)$, $B = (a, 0)$, $C = (a, b)$, and a, b are unknown positive constants. Define $(U_i, V_i) = (X_i, Y_i/X_i)$. Show that, for each $i = 1, ..., n$, the random variable U_i is independent

of V_i. Hence or otherwise, find the maximum likelihood estimator of the area of the triangle ABC.

6. Let $Y = e^{UX} + Z$, where U, X, Z are independent. Here X is uniformly distributed over $(0, 1)$, Z has the standard normal distribution and U has density $f(u) = 2u$, for $0 < u < 1$.

   (a) Find the best predictor of Y given $X = \frac{1}{2}$ when U and Z are unknown.

   (b) Find the best *linear* predictor of Y given $X = \frac{1}{2}$ when U and Z are unknown.

7. Suppose that you have a random observation X with density f over the real line. Based on X, we want to test

$$H_0 : f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ versus } H_1 : f(x) = \frac{1}{\pi(1 + x^2)}.$$

   (a) Prove that a test which rejects H_0 if and only if $|X| > \Phi^{-1}(0.975)$ is a most powerful test of level $\alpha = 0.05$ for this problem. Here $\Phi(x)$ is the standard normal distribution function.

   (b) Find the power of the test described in part (a) above.

8. Consider a completely randomised design (CRD) with t treatments, $t \geq 2$, and $n = 4t + 3$ observations. If we want to minimise the average variance of the best linear unbiased estimators (BLUEs) of all the pairwise comparisons of treatment effects, how many observations should be allocated to each treatment?