2015

BOOKLET NO. TEST CODE: **STA**

Forenoon

No. of Questions: 10 **Time: 2 hours**

- **Answer as many questions as you can.**
- **Do not feel discouraged if you are not able to answer all the questions.**
- **Partial credit may be given for partial answer.**
- **Full credit will be given for complete and rigorous arguments.**

Write your Name, Registration Number, Test Code, Booklet No. etc., in the appropriate places on the answer-booklet.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER-BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP! WAIT FOR THE SIGNAL TO START.

1. Suppose $t > 0$ is such that $e^x > x^t$ for all $x > 0$. Show that t must be less than e.

2. For $n \geq 1$, let

$$g_n \stackrel{def}{=} \text{geometric mean of } \{1/n, 2/n, \ldots, n/n\}.$$

Find $\lim_{n\to\infty} g_n$.

3. Let $\mathbf{X}$ be a 6×3 matrix and $\boldsymbol{y} \in \mathbb{R}^6$ be given by

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix}.$$

   (a) Find the vector $\boldsymbol{z}_1 \in \mathbb{R}^6$ which lies in the vector space spanned by the first column of $\mathbf{X}$ and is closest to $\boldsymbol{y}$ in terms of minimum Euclidean distance. Justify your answer.

   (b) Find the vector $\boldsymbol{z}_2 \in \mathbb{R}^6$ which lies in the column space $\mathcal{C}(\mathbf{X})$ of $\mathbf{X}$ and is closest to $\boldsymbol{y}$ in terms of minimum Euclidean distance. Justify your answer.

4. Using probabilistic arguments or otherwise, prove that

$$\sum_{k=m+1}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} \leq \sum_{j=0}^{n-1} \binom{m}{j} p^j (1-p)^{m-j}$$

for $0 < p < 1$ and $m > n > 0$.

5. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having Uniform$(-1, 1)$ distribution. Let

$$Y = X_i \text{ if } \mid X_i \mid \leq \mid X_j \mid \text{ for all } j \neq i.$$

Find $\mathrm{E}(Y)$.

6. Suppose that $\boldsymbol{X}$ follows a bivariate normal distribution with mean vector $\mathbf{0}$ and dispersion matrix $\boldsymbol{\Sigma} = \mathrm{diag}(1, 1)$. Define $Y_1 = X_1\, sign(X_2)$ and $Y_2 = X_2\, sign(X_1)$.

(a) Find the marginal distribution of Y_1.

(b) Show that (Y_1, Y_2) cannot have a bivariate normal distribution.

(c) Find the correlation coefficient between Y_1 and Y_2.

7. Let T_1 and T_2 be two independent random variables such that $T_1 \sim N(\theta, \sigma_1^2)$ and $T_2 \sim N(2\theta, \sigma_2^2)$, where $-\infty < \theta < \infty$ is an unknown parameter and σ_1^2, σ_2^2 are known, and $\sigma_1^2 \neq \sigma_2^2$.

   (a) Find a complete sufficient statistic for θ.

   (b) Hence, or otherwise, find the uniformly minimum variance unbiased estimator (UMVUE) of θ based on T_1 and T_2.

8. Let $X_1, \ldots, X_n$ be a random sample from Uniform$(0, \theta)$ and assume a Pareto prior for θ with probability density function

$$\pi(\theta) = \frac{\beta}{\alpha}\left(\frac{\alpha}{\theta}\right)^{\beta+1}, \quad \alpha < \theta < \infty.$$

   Find the posterior mode of θ.

9. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed observations from Uniform$(0, \theta)$ distribution, $\theta > 0$. Let $X_{(n)} \stackrel{def}{=} \max(X_1, \ldots, X_n)$. We wish to test the hypothesis $\mathrm{H}_0 : \theta \leq 1/2$ versus $\mathrm{H}_1 : \theta > 1/2$. Consider the following test statistic:

$$\delta_c = \begin{cases} 1 & \text{if } X_{(n)} \geq c, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Obtain the expression for the power function of δ_c and show that it is a monotone increasing function of θ. j

   (b) What choice of c would make δ_c have size exactly 0.05? (This choice will depend on n.)

   (c) Obtain an expression for the smallest possible n so that δ_c obtained in (b) has power 0.98 for $\theta = 3/4$?

10. Consider a confounded 2^5 factorial experiment, with factors A, B, C, D, E, arranged in 4 blocks of 8 plots each. If one of the blocks consists of the 8 treatment combinations $b, ad, ae, abcd$, $abce$, c, bde, cde, then

(a) Write down the principal block.

(b) Identify the confounded factorial effects.

(c) Write down the expression of the contrast representing interaction AB.