2016

BOOKLET No. **TEST CODE: PQB**

Afternoon

Time: 2 hours

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On the answer-booklet write your Registration Number, Test Code, Number of this Booklet, etc. in the appropriate places.

There are two parts in this booklet as detailed above.

Candidates having **'Statistics/Mathematics'** background are required to answer questions from **Part I** only as per instructions given.

Candidates having **'Engineering'** background are required to answer questions from **Part II** only as per instructions given.

USE OF CALCULATOR IS NOT ALLOWED.

STOP! WAIT FOR THE SIGNAL TO START

PART I (for STATISTICS / MATHEMATICS STREAM)

ATTENTION: ANSWER A TOTAL OF SIX QUESTIONS, TAKING AT LEAST TWO FROM EACH GROUP.

GROUP S1: Statistics

1. Let $X_1, X_2, \cdots, X_n$ be a random sample from the population with probability density function

$$f(x) = \begin{cases} 3\lambda x^2 e^{-\lambda x^3} & \text{if } x > 0,\ \lambda > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Derive the maximum likelihood estimate of λ, and hence find an unbiased estimator of λ.

(b) Derive an exact $100(1-\alpha)\%$ confidence interval of λ.

[(4+10) + 6 = 20]

2. A random sample of size 3 is drawn from a population of size N with replacement. Obtain the probabilities that the sample contains 1, 2 and 3 distinct units. If $\bar{y}_d$ denotes the unweighted mean over the distinct units in the sample, show that the variance of $\bar{y}_d$ is

$$\operatorname{Var}(\bar{y}_d) = \frac{(2N-1)(N-1)S^2}{6N^2}.$$

Hence, show that $\operatorname{Var}(\bar{y}_d) < \operatorname{Var}(\bar{y})$, where
$\bar{y}$ is the ordinary mean of the 3 observations in the sample;
$S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$ is the population variance;
y_i = ith observation in the population, and
$\bar{Y}$ = population mean.

[6+10+4 = 20]

3. Management of ABC Fans Limited suspected that background music may improve assembly line productivity of table fans. An industrial engineer is tasked to carry out an experiment to verify the suspicion. The engineer selects four different music types (Light Hindustani, Light Carnatic, Indian Folk and Hindi Filmi) to study their effects on the productivity. Four different days of a week are selected for study. It is generally believed that assembly method produces such fatigue that the time required for the last assembly may be greater than the time required for the first. To account for this source of variability, four different assembly time periods, of one hour each, are considered for experimentation. Now answer the following questions:

(i) To plan this experiment, what type of design do you propose? Give a design layout for conducting the experiment. Identify the treatments and the block factors.

(ii) Under Gauss-Markov set up, write the linear model for your proposed design, describing the model parameters. (State all the assumptions associated with the model.)

(iii) Derive the normal equations.

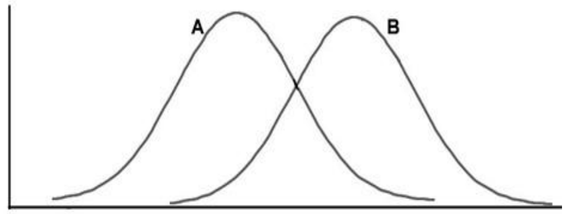
(iv) Are all the parametric functions of model parameters estimable? If yes, why? And if not, what type of functions are estimable? (No explicit derivation is required.)

[(2+2+1) + 3 + 8 + 4 = 20]

4. (a) Consider a population with N members labelled consecutively from 1 to N. Let $X_1, X_2, \cdots, X_n$ be a random sample drawn with replacement from the population. Find a sufficient statistic for N.

(b) A company procures material from two different suppliers A and B. The lead time to supply are random variables, say, X_A and X_B respectively. Data on lead times are collected for both the suppliers and frequency distributions are constructed.

Rough sketch of the frequency density functions is given below:



(i) Give the sketch of the ogives.

(ii) Use the ogives to assess which supplier would you prefer. Justify.

Note that a supplier is preferable if its lead time is more likely to be smaller, merely having a lower mean lead time is not sufficient.

(c) Suppose we have two independent samples

$$\{(y_i, x_i), \quad i = 1,2,\cdots,n_1\}$$

and

$$\{(y_i, x_i), \quad i = n_1 + 1,\cdots,n_1 + n_2\}$$

to fit two regression models:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1,\cdots,n_1$$

and

$$y_i = \gamma_0 + \gamma_1 x_i + \epsilon_i, \quad i = n_1 + 1,\cdots,n_1 + n_2$$

respectively, where ϵ_i's are independent and identically distributed $N(0, \sigma^2)$ variables.

Combine these two models as a single regression model. Hence, derive a test for

$$\mathrm{H}_0: \beta_1 = \gamma_1 \text{ vs. } \mathrm{H}_1: \beta_1 \neq \gamma_1.$$

(d) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n drawn from a population with distribution function F. The corresponding empirical distribution function is defined as

$$F_n(x) = \frac{\text{number of } X_i \text{ 's in the sample } \leq x}{n}.$$

Find the variance of $F_n(x)$ for fixed x.

[4 + (3+3) + (2+4) + 4 = 20]

5. **(a)** In a cable manufacturing industry, input resistance (X) and output resistance (Y) are two important parameters of a cable. An engineer claims that the variance of Y is twice the variance of X. To verify his claim, n pairs of observations on (X, Y) are collected. Perform a suitable test to validate the engineer's claim based on the data. (State the necessary assumptions you make.)

(b) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a population having probability density function:

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0,\ \theta > 0.$$

(i) Derive a most powerful level α test for
$$\mathrm{H}_0\colon \theta = \theta_0 \text{ vs. } \mathrm{H}_1\colon \theta = \theta_1 > \theta_0.$$

(ii) Show that the test derived in **(i)** above is also a UMP level α test for $\mathrm{H}_0\colon \theta = \theta_0$ vs. $\mathrm{H}_1\colon \theta > \theta_0$.

[8 + (9+3) = 20]

GROUP E2: Probability

6. **(a)** An envelope is known to have come by post from either TATANAGAR or KOLKATA. Only two consecutive letters 'TA' are visible on the envelope. What is the probability that the actual word is KOLKATA?

(b) A man parks his car among the $(n-1)$ cars already parked in a row, $n \geq 3$. His car is not parked at any end. On his return, he finds that exactly m out of total n cars are still there. What is the probability that both the cars parked on the two sides of his car have left?

(c) Suppose that $Z \sim N(0,1)$. Find $E[Z^2\Phi(Z)]$, where $\Phi(\cdot)$ is the distribution function of Z.

[6 + 6 + 8 = 20]

7. **(a)** Consider the quadratic equation: $2x^2 + Bx + 5 = 0$, where B is a random variable having the following distribution

$$B = \begin{cases} -10 \text{ with probability } \frac{1}{3}, \\ 3 \quad\quad \text{with probability } \frac{2}{3}. \end{cases}$$

What is the probability that both the roots of the equation are real? Find the expectation of the sum of the roots.

(b) Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order-statistics corresponding to n independent and identically distributed (i.i.d.) random variables $X_1, X_2, \cdots, X_n$ having common probability density function $(x) = \lambda e^{-\lambda x}$, $x > 0, \lambda > 0$. Define $Z_1 = nX_{(1)}$ and $Z_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$, $i = 2, \cdots, r$, where $r \leq n$. Show that $Z_1, Z_2, \cdots, Z_r$ are i.i.d. exponential random variables.

[(3+3) + 14 = 20]

8. **(a)** Assume that $X_1, X_2, \cdots, X_n$ are independent and identically distributed random variables having common distribution as $N(\mu, 1)$. Define

$$S_n = \sum_{i=1}^{n} X_i^2 \text{ and } T = aY_k$$

where $Y_k \sim \chi_k^2$, a and k are suitable constants.

Find the values of a and k so that the random variables S_n and T have the same mean and variance.

(b) Suppose $X \sim N(0,1)$ and $Y|X \sim N(1 - X, 1)$. Find the correlation coefficient between X and Y.

[10 + 10 = 20]

9. **(a)** A fair die is tossed repeatedly and the outcome is recorded. What is the probability that we need exactly 10 tosses to get four 6s?

(b) Consider the population of 5×7 matrices having each entry as 1 or 2. A matrix is randomly selected from the population. What is the probability that the sum of the entries in each row and each column of the selected matrix is odd?

[8 + 12 = 20]

10. **(a)** Examine whether the weak law of large numbers holds for the sequence $\{X_k\}_{k \geq 1}$ of independent random variables with probability distribution given below:

$$\mathrm{P}(X_k = \pm 2^k) = 2^{-(2k+1)},$$
$$\mathrm{P}(X_k = 0) = 1 - 2^{-2k}.$$

(b) A flea moves from one vertex to another of a triangle in the following way. From vertex i, it moves to the clockwise-neighbour vertex with probability p_i and to the anticlockwise-neighbour vertex with probability $1 - p_i$, $i = 1,2,3$. Find, in terms of p_i's, the proportion of time that the flea is at each of the vertices.

[12 + 8 = 20]

PART II (for ENGINEERING STREAM)

ATTENTION: ANSWER A TOTAL OF SIX QUESTIONS TAKING AT LEAST TWO FROM EACH GROUP.

GROUP E1: Mathematics

1. (a) A teacher coaches for the Joint Entrance Examinations. In her tutorial, 35 students are studying Mathematics, 30 Physics, 25 Chemistry, 17 Mathematics and Physics, 15 Mathematics and Chemistry, 12 Physics and Chemistry, and 10 students are studying all the three subjects. She charges Rs. 300/- from each student per month for a subject. What is her monthly earning from the tutorial? [No credit will be given for deriving the result using Venn diagram.]

(b) If $a, b,$ and c are three sides of a triangle which is not an equilateral triangle then show that

$$\frac{1}{(a+b-c)}+\frac{1}{(b+c-a)}+\frac{1}{(c+a-b)}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\,.$$

[10+10=20]

2. (a) Find the maximum value of the function $f(x) = x^{1/x},\ x > 0.$

(b) Hence, or otherwise, and without explicitly evaluating show that

$$e^{\pi} - \pi^{e} > 0.$$

(c) The three sides of a right-angled triangle form a geometric progression. Find the tangents of the two acute angles.

[8+5+7=20]

3. (a) Let ABC be a right-angled triangle, with $\angle A = 90°$. Suppose that the lengths of the sides containing the right angle are b and c. Let AD be the bisector of the right angle, meeting BC at D. Find the length of AD.

(b) Prove that there is no term independent of x in the expansion of

$$\left(\sqrt{\frac{3}{2x^2}}+\sqrt{\frac{x}{3}}\right)^{10}.$$

(c) Find the volume of the solid whose base is in the region on the xy-plane bounded by the parabola $y = 6 - x^2$ and the line $y = 5x$ and where the top is bounded by the plane $z = x + 7$.

[8+4+8=20]

4. (a) The sides of a triangle are in arithmetic progression and its area is $\frac{3}{5}$th of the area of an equilateral triangle with the same perimeter. Prove that, the ratio of the sides is either 3:5:7 or 7:5:3.

(b) Find the integral

$$I = \int_1^a [x^2]\, f'(x)dx,$$

where $a > 1$ is a positive real number, and $[x]$ means the greatest integer less than or equal to x.

[10+10=20]

GROUP E2: Engineering & Technology

Engineering Mechanics and Thermodynamics

5. (a) A rod of square section of side *D* at one end tapers to a square section of side *d* at the other end. The rod is of length *L* and is subjected to an axial pull *P*. Find the increase in length of the rod.

(b) A train of weight 1500 kN is ascending (i.e. going upwards) a slope of 1 in 100 with a uniform speed of 36 km/hr. Find the power exerted by the engine, if the load resistance is 5 N per kN weight of the train.

(c) A wagon weighing 90 kN moving at 18 kmph strikes a pair of buffer springs. If the stiffness of each spring is 600 kN/m, determine the maximum compressive force acting on each spring.

[6+7+7=20]

6. **(a)** Calculate the moment of inertia I of a right circular cone of uniform density, radius of base 'a' and height 'h' with respect to its geometric axis.

(b) A reversible engine operates between temperatures T_1 and T $(T_1 > T)$. The energy rejected from this engine is received by a second reversible engine at the same temperature T. The second engine rejects energy at temperature T_2 $(T_2 < T)$. Establish relation between T_1, T_2 and T under the following conditions:

(i) when the two engines produce the same amount of work output;

(ii) when the two engines have the same cycle efficiencies.

[10+(5+5)=20]

7. **(a)** Show that the air standard efficiency for a cycle comprising two constant pressure processes and two isothermal processes (all reversible) is given by

$$\eta = \frac{(T_2 - T_1)\ln(x)^{\frac{\gamma-1}{\gamma}}}{T_2\left[1 + \ln(x)^{\frac{\gamma-1}{\gamma}} - T_1\right]}$$

where T_2 and T_1 are the maximum and minimum temperatures of the cycle, and x is the pressure ratio.

(b) A body of constant heat capacity C and at temperature T_1 is put in contact with the reservoir at a higher temperature T_2.

The pressure remains constant while the body comes to equilibrium with the reservoir. Show that the entropy change of the universe is equal to

$$C\left[\frac{T_1 - T_2}{T_1} - \ln\left(1 + \frac{T_1 - T_2}{T_1}\right)\right].$$

[10+10=20]

Electrical and Electronics Engineering

8. **(a)** A 100 kVA transformer has 400 turns on the primary and 80 turns on the secondary. The primary and secondary resistances are 0.3 Ω and 0.01 Ω respectively and the corresponding leakage reactances are 1.1 Ω and 0.035 Ω. Calculate the equivalent impedance referred to the primary circuit.

(b) The power input to a three-phase induction motor is 50 kW and the corresponding stator losses are 2 kW. Calculate

(i) the rotor I^2R loss when the slip is 3%;

(ii) the total mechanical power developed neglecting rotor core loss which is very small at 3% slip;

(iii) the output power of motor if the friction and windage losses are 1 kW;

(iv) the efficiency of the motor.

(c) A dc generator is connected to a 220 V dc mains. The current delivered by the generator to the mains is 100 A. The armature resistance is 0.1 Ω. The generator is driven at a speed of 400 rpm. Calculate

(i) the induced emf;

(ii) the electromagnetic torque.

[5+(3+3+2+2)+(2+3)=20]

9. **(a)** Calculate the average and r.m.s. value of current represented in the Fig. 1 shown below:

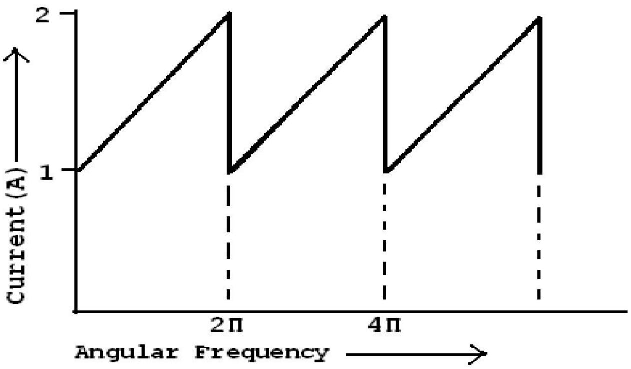


Fig. 1

(b) For the circuit, shown in Fig. 2, $V_1 = 4V$, $V_2 = 8V$, $R_1 = 8\Omega$, and $R_2 = 6\Omega$. Find the Thevenin equivalent for the network to the left of terminals a, b. Assume that the internal resistances of the batteries are 0.

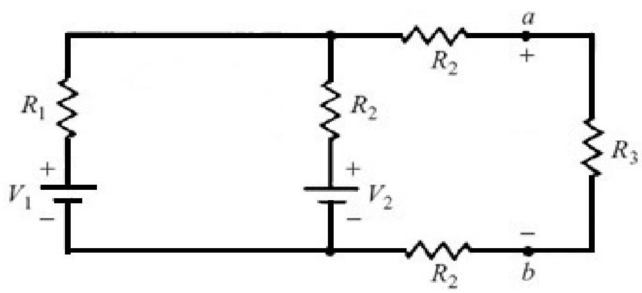


Fig. 2

(c) For the circuit, shown in Fig. 3, find the output voltage v_0 for $v_S = 1.0$V. Assume, diode D_2 is ideal, $R_1 = 2$ kΩ, $R_2 = 8$ kΩ, and $R_3 = 8$ kΩ.

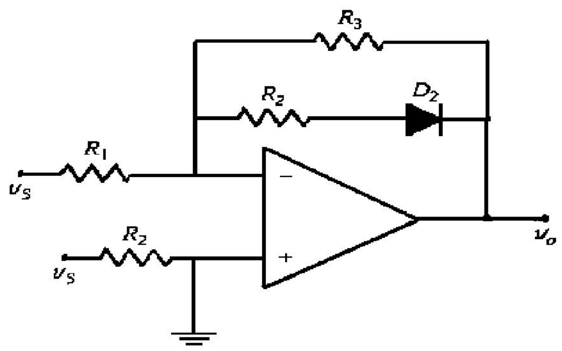


Fig. 3

[(4+4)+7+5=20]

Engineering Drawing

10. (a) Sketch the top view, front view and right side view of the object shown in Fig. 4.

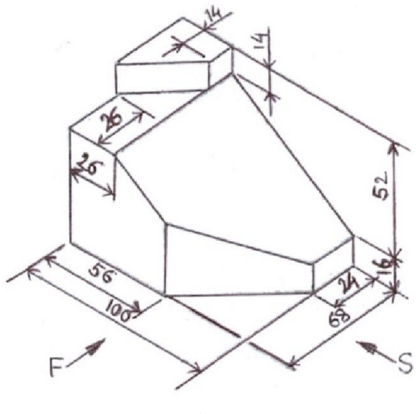


Fig. 4

(b) Draw or sketch the (i) front view, (ii) side view from the left and (iii) top view of the given object in Fig. 5.

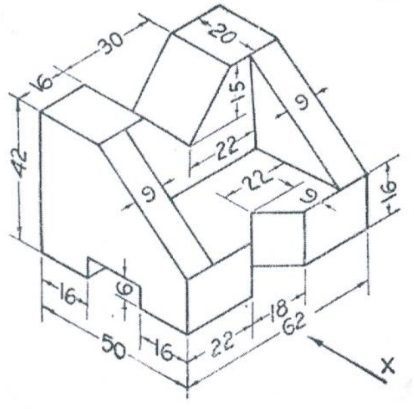


Fig. 5

[10+10=20]