2016

BOOKLET No. TEST CODE: **PCB**

Afternoon

TIME: 2 HOURS	
GROUP	MAX SCORE
A	**30**
B	**70**

Write your Registration Number, Test Centre, Test Code and the Booklet No. in the appropriate places in the answer-book.

The questions are divided into *two* groups, A and B.

- Answer **ALL** questions in **GROUP A**.
- **GROUP B** consists of the following ***FIVE*** sections:

  I. **Computer Science**
  II. **Engineering and Technology**
  III. **Mathematics**
  IV. **Physics**
  V. **Statistics**

  Answer questions from **ONLY ONE SECTION** in **GROUP B**.

The marks allotted to each question appear within the brackets [ ] following it.

YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP! WAIT FOR THE SIGNAL TO START.

Group A

Answer *ALL* questions.

A1. If α, β, γ are the roots of the equation $x^3+6x+1=0$, then prove that

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{\beta}{\gamma}+\frac{\gamma}{\beta}+\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}=-3.$$

[8]

A2. Let n be a fixed positive integer. For any real number x, if for some integer q,

$$x = qn + r, \quad 0 \leq r < n,$$

then we define $x \bmod n = r$.

Specify the points of discontinuity of the function $f(x) = x \bmod 3$ with proper reasoning. [10]

A3. A bit string is called legitimate if it contains no consecutive zeros, e.g., 0101110 is legitimate, whereas 10100111 is not. Let a_n denote the number of legitimate bit strings of length n. Define $a_0 = 1$. Derive a recurrence relation for a_n (i.e., express a_n in terms of the preceding a_i's). [12]

Group B

Section I : Computer Science

Answer any *FIVE* questions.

C1. (i) Consider the array $A = [20, 13, 19, 8, 3, 5, 4]$ that represents a heap. Draw the heap after removing the element 20.

(ii) List all distinct integer keys k such that, when k is inserted in the Binary Search Tree of Figure 1, its height increases. Note that you are not allowed to insert an already existing key again. Justify your answer.

[5 + (3+6) = 14]

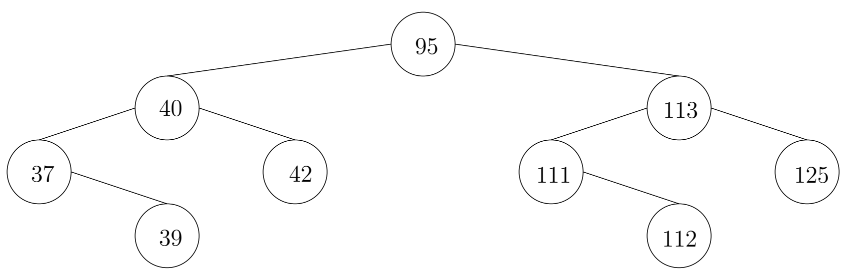

Figure 1: The given Binary Search Tree

C2. (i) Consider sending a large file of 360,000 bits from Host A to Host B, connected through a router, as shown in Figure 2. Assume that there is no queuing and propagation delay, and the router has sufficient buffer space. Host A splits the file into segments of S bits each and adds 36 bits of header to each segment, forming packets of $(36 + S)$ bits. Each link has a transmission rate of R bps. Find the value of S that minimizes the time needed to move the file from Host A to Host B.

(ii) The CPU of a system having an execution rate of 1 million instructions per second needs 4 machine cycles on an average for executing an instruction. On an average, 50% of the cycles use memory bus. For execution of the programs, the

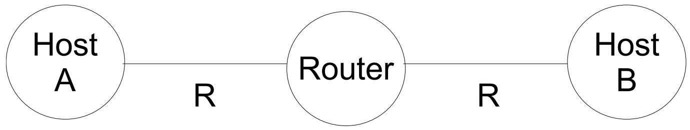

Figure 2: Links and Router between Host A and Host B

system utilizes 90% of the CPU time. For block data transfer, an I/O device is attached to the system, while the CPU executes background programs continuously. Determine the maximum I/O transfer rate for each of the two cases: (a) programmed I/O, (b) cycle-stealing DMA (in transparent mode). You may assume that transferring one byte involves 4 operations: `in-status`, `check-status`, `branch` and `read/write` in memory, each requiring one machine cycle.

[7+ (3+4) = 14]

C3. Consider the following extract from a program, written in a C-like language, that computes the transpose of a matrix.

```
for (i = 0; i < N; i++)
    for (j = 0; j < N; j++)
        B[i,j] = A[j,i];
```

`A` and `B` are `N`×`N` matrices with floating point entries that are stored in memory in row-major order as shown in the example below.

A[0,0]	A[0,1]	A[0,2]	...	A[0,N-1]	A[1,0]	A[1,1]	...	A[N-1,N-1]

This program runs under an OS that uses demand-paging. Considering **only** memory references to the matrix entries, and using the information given below, compute the page fault rate for the matrix transposition code given above.

- Page size: 2^{10} bytes
- Number of frames allocated to the program: 8
- Page replacement policy: LRU
- `N` $= 2^{16}$
- Size of each matrix entry: 8 bytes

- Each of `A` and `B` is stored starting from the beginning of a page
- None of the pages allocated to `A` or `B` are initially in memory

[14]

C4. Commodity items have some positive or negative changes in their prices each week. Each trading company picks a portfolio of commodity items, that is, they have one or more items and they own some non-zero quantity of each one. The database table for this problem consists of the following two relations:

Commodity(item-number *int*, price-change *float*, week *int*),
Owns(company-name *text*, item-number *int*, quantity *float*).

Write an SQL query which returns the item-numbers of commodities for which, in any given week, the price change is greater than or equal to that of all other items and there exists at least one company selling that item only (i.e., not selling any other item) in that week. [14]

C5. What will be the output of the following C program? If you think it will give a runtime error, you need to mention it. In either case, your answer must include proper justifications without which no credit will be given.

```
#include<stdio.h>
main()
{
        unsigned char i, j, a[] = {1, 2, 3, 4, 5};
        int n;
        i = j = n = 5;
        while(i-- != 0) n += a[i];
        while(j++ != 0) n += 2;
        printf("i = %d, j = %d, n = %d\n", i, j, n);
        while(j-- != 0) a[0] += n;
        printf("j = %d, a[0] = %d\n", j, a[0]);
}
```

[14]

C6. (i) Let L be a regular language over $\{0, 1\}$. Define the reverse of the language L to be the language

$$L^R = \{w \in \{0,1\}^* \;:\; \mathit{reverse}(w) \in L\},$$

where $reverse(w)$ denotes the string w read in reverse. For example $reverse(0001) = 1000$. Show that L^R is regular.

(ii) Let $L = \{x \ : \ x \in \{0,1\}^*$, x contains an odd number of 1's and 00 as a substring$\}$. Construct a regular expression for the language L.

[6+8 = 14]

C7. Let A be a sorted array of n positive integers, such that $A[1] \leq A[2] \leq \ldots \leq A[n]$. Given an integer x as input, the goal is to find two array indices k and l such that $A[k] + A[l] = x$, if such indices exist; otherwise, the goal is to report "Failure". Design an algorithm for this problem, that works in $O(n)$ time. [14]

C8. (i) Consider all possible trees with n nodes. Let k be the number of nodes with degree greater than 1 in a given tree. What is the maximum possible value of k? Justify your answer.

(ii) Consider $2n$ committees, each having at least $2n$ persons, formed from a group of $4n$ persons. Prove that there exists at least one person who belongs to at least n committees.

[(1+4) + 9 = 14]

Section II : Engineering and Technology

Answer any *FIVE* questions.

E1. (i) Identical coils X and Y have 200 turns each and lie in parallel planes such that 60% of the flux produced by one coil links with the other. A current of 1 A in coil X produces a flux of 0.01 milliWeber. Calculate

(a) the self-inductance of each coil and their mutual inductance, and

(b) the magnitude of the emf induced in coil Y if the current in coil X changes from 0 to 15 A in 0.01 seconds.

(ii) Consider a parallel plate capacitor with each plate having an area of 10 cm^2. The plates are placed 1 mm apart and filled with a dielectric of relative permittivity 10. The capacitor is charged to 100 V. Subsequently, the plates are isolated from the supply and the dielectric material is removed. Calculate the voltage between the plates now. Also calculate the voltage when the plates are moved to a distance of 2 mm apart.

[((2+2)+5)+(3+2)=14]

E2. (i) Given two integers x, y, $0 \leq x, y \leq 7$, design a combinational logic circuit to compute the function $Z = (-1)^x \cdot x + (-1)^y \cdot y$. Suppose that integers are represented in 2's complement form. Use only full adder blocks and EX-OR gates in the design.

(ii) In a 32-bit machine, floating point numbers are represented in the following format:

S	E	F
1 bit	8 bits	23 bits

Figure 3: Figure for E2.(ii)

In the normalized form, the value of a number is

$$V = (-1)^S \times 2^{E-127} \times 1.F$$

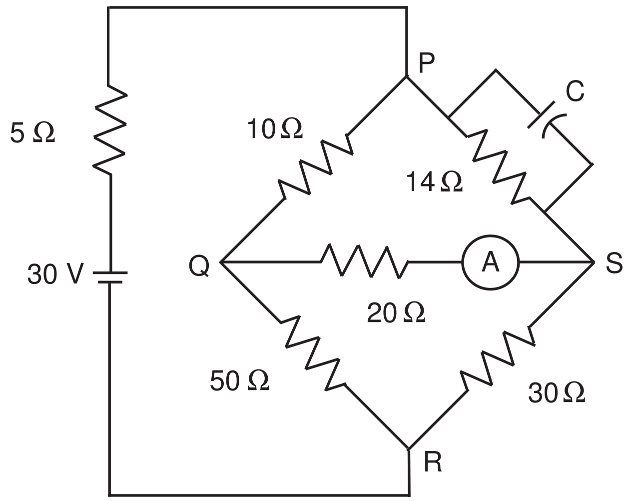

Figure 4: Circuit for Question E3.(i)

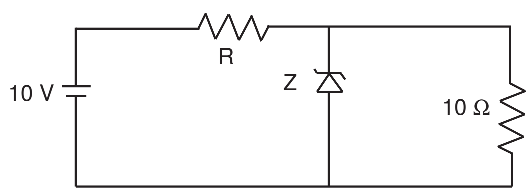

Figure 5: Circuit for Question E3.(ii)

If a floating point number represented in this format is given as 40 60 00 00 in hexadecimal, what is its decimal value?

[10+4=14]

E3. (i) Consider the circuit shown in Figure 4 where an ideal dc ammeter (marked as A) has been connected to measure the current flowing from Q to S. Calculate the steady-state current shown by the ammeter and also the steady-state voltage across the capacitor C.

(ii) In the circuit shown in Figure 5, Z is a Zener diode with a breakdown voltage of 5 V having maximum power dissipation capacity of 1.5 W. Calculate the minimum and maximum values of R so that the Zener diode operates in its breakdown region.

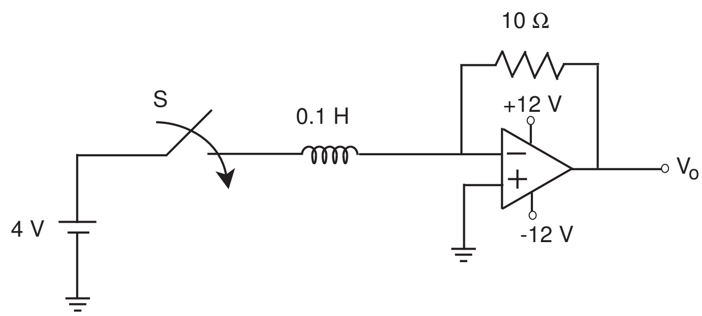

Figure 6: Figure for Question E6.(i)

[7+7=14]

E4. Design a sequence detector which produces an output 1 every time the sequence 0101 is detected, the rightmost "1" of the sequence 0101 being the current bit in the sequence. An output 0 is generated when any other sequence is detected. For example, given an input sequence . . . 001010100, the output sequence will be . . . 00001010.
Show the state diagram and design the circuit using D flip-flops and elementary logic gates. [4+10=14]

E5. (i) A U-tube held vertically, contains a non-viscous liquid. It is now accelerated horizontally with a constant acceleration of 2 m/s^2 in the same plane containing the two vertical limbs. If the separation between the vertical limbs is 10 cm, find the difference in heights of the liquid in the two limbs due to the acceleration of the U-tube. (Assume g=10 m/s^2 and neglect surface tension of the liquid).

(ii) A rod of length 3 m, cross-sectional area 2 cm^2 and mass 3 kg is whirled in a horizontal circle with constant angular velocity of 6 rad/sec about one of its ends. Find the extension in the length due to rotation if the Young's modulus of the material of the rod is 2×10^{11} N/m^2.

[6+8=14]

E6. (i) Refer to the circuit shown in Figure 6, where S is a switch which closes at time $t = 0$. Derive an expression for the output voltage V_0 as a function of time t and sketch its waveform from $t = 0$ to $t = 1$ s.

(ii) A sinusoidal voltage of 10 volts rms value at 5 kHz is applied to the series combination of a 10 Ω resistance, $\frac{0.01}{\pi}$ Henry inductance and $\frac{1}{\pi}$ μF capacitance. Calculate the rms current flowing through the circuit. Further, determine the rms voltage across the inductor, capacitor and resistor separately and also the total voltage drop across all these three circuit elements.

[(4+2)+(2+3+3) = 14]

E7. (i) A small block of super-dense material has a mass of 2×10^{24} kg. It is placed at a height h above the earth's surface from where it is allowed to fall towards the earth's surface under the effect of mutual gravitational force between the block and the earth. Find the speed of the block as a function of h when its height from the earth's surface has reduced to $\frac{h}{2}$. The mass of the earth is 6×10^{24} kg. Assume that $h \ll R$, R being the radius of the earth.

(ii) A mechanic pushes a car weighing 250 kg with a force of 200 N over a distance of 10 m on a rough road. Then he gets tired and his applied force decreases linearly (with distance) to 125 N. The total distance the mechanic moves is 25 m. Calculate the work done by the force applied by the mechanic and the total work done by the frictional force on the car, considering the coefficient of friction to be 0.05 and the gravitational acceleration to be 10 ms^{-2}.

[7+(4+3)=14]

E8. A 3-phase 50 Hz Δ-Y connected transformer is supplying to a Y-connected balanced load. Let its maximum efficiency be achieved at three-fourth of the full-load. Determine the full-load efficiency of the transformer at power factor 0.8, when the input line voltage is 2.2 kV and the output phase current is 10 A. Assume a turns ratio of 1:5 and the iron loss to be 3.6 kW. [14]

E9. A 12-pole, 3-phase alternator driven at a speed of 500 rpm supplies $200\sqrt{3}$ V to an 8-pole, 3-phase Y-connected induction motor. The rotor is also Y-connected with stator to rotor turns ratio of 2:1. Assume that the rotor resistance per phase is 0.6 Ω and the standstill rotor reactance per phase is 20 Ω.

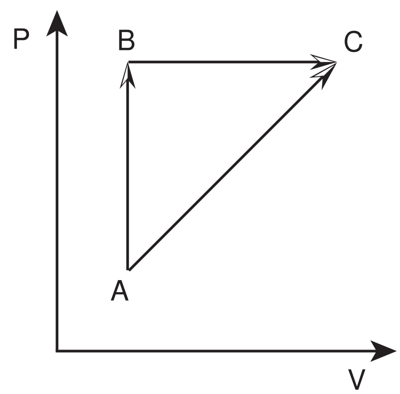

Figure 7: Sketch for Question E10.(i)

(a) Calculate the speed of the motor if slip is 4%.

(b) Calculate the rotor current.

(c) If the motor speed decreases to 705 rpm due to extra load, calculate the frequency of the rotor emf.

[4+6+4=14]

E10. (i) A thermodynamic process is shown in Figure 7. $P_A = 2 \times 10^4$ Pa, $V_A = 6 \times 10^{-3}$ m^3, $P_B = 5 \times 10^4$ Pa and $V_C = 8 \times 10^{-3}$ m^3. During the transitions along AB and BC, 400 J and 800 J of heat are added to the system, respectively. Find the change in internal energy of the system when it moves from A to C.

(ii) An ideal gas having initial pressure P, volume V and temperature T is allowed to expand adiabatically until its volume becomes $8V$ while its temperature falls to $\frac{1}{2}T$.

(a) Determine the degrees of freedom of the gas molecules.

(b) Obtain the work done by the gas during the expansion as a function of the initial pressure and volume.

[7+(3+4)=14]

E11. The following C function is supposed to print an isosceles trapezium using the * character. The trapezium should be oriented such that its parallel sides are printed vertically on the screen and the number of $*$'s printed on a non-parallel side is the same as the maximum number of $*$'s printed on a row (i.e., a horizontal line). For example, `trapezium(8, 4)` should print the following output:

```
*
* *
* * *
* * *
* * *
* * *
* *
*
```

You may assume that in the function, the parameters `longer_side` and `shorter_side` are both even, and `longer_side` > `shorter_side`.

```
void trapezium(int longer_side, int shorter_side)
{
    int height = _____ ;
    int j, k;
    for (j = __ ; j _____ ; j__) {
        for (k = __ ; k _____ ; k__) putchar('*');
        putchar('\n');
    }
    for (j = __ ; j _____ ; j__) {
        for (k = __ ; k _____ ; k__) putchar('*');
        putchar('\n');
    }
    for (j = __ ; j _____ ; j__) {
        for (k = __ ; k _____ ; k__) putchar('*');
        putchar('\n');
    }
    return;
}
```

Fill in the blanks in the body of the function so that it fulfills its intended purpose. [14]

E12. (i) A solid sphere with a constant speed of 5 m/s is rolling without slipping on a level surface. Determine how far it can roll up a 30° inclined plane before it comes to rest. The moment of inertia of a solid sphere about its diameter is $2mr^2/5$ where m is the mass of the solid sphere and r is its radius. Assume g=10 m/s^2.

(ii) An impulse J is exerted horizontally on a solid sphere of radius R placed on a frictionless horizontal surface. The line of action of the impulse is at a height h above the center of the sphere. If the rotational and translational kinetic energies of the sphere just after the impulse are equal, then find the value of h in terms of R.

[8 + 6 = 14]

Section III : Mathematics

Answer any *FIVE* questions.

M1. Let N be a normal subgroup of G such that the order of N is n and the order of the quotient group G/N is m. If the greatest common divisor of m and n is 1, show that N is the only subgroup of G of order n. [14]

M2. Let $\mathbb{R}$ denote the field of real numbers, and $\mathbb{R}[X]$ the polynomial ring in one variable. Let $I := (X^3 + X^2, X^5)\mathbb{R}[X]$ be the ideal of $\mathbb{R}[X]$ generated by $X^3 + X^2$ and X^5.

(a) Find $f(X) \in \mathbb{R}[X]$ such that $I = f(X)\mathbb{R}[X]$.

(b) Prove that I is contained in exactly one prime ideal of $\mathbb{R}[X]$.

[7+7=14]

M3. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, where $\mathbb{R}$ is the set of real numbers. Suppose that there exists a sequence $\{a_n : n \geq 1\}$ of non-zero real numbers such that

(1) $\{a_n\}$ converges to zero, and

(2) $f(a_n) = 0$ for all n.

Show that $f'(0) = 0$.

(ii) Solve the differential equation

$$dx + dy = \frac{x+y}{x^2+y^2}\left(xdy - ydx\right).$$

[8+6=14]

M4. Let $x_1 = 1$, $x_2 = 2$, and for $n = 1, 2, \ldots$, define

$$x_{2n+1} = \frac{5x_{2n-1} + 2x_{2n}}{7}, \quad x_{2n+2} = \frac{2x_{2n-1} + 5x_{2n}}{7}.$$

Show that $\lim_{n\to\infty} x_n$ exists and find the limit. [10+4=14]

M5. Let $\mathbb{Q}$ denote the field of rational numbers and A denote an integral domain containing $\mathbb{Q}$. Suppose that the dimension of A as a vector space over $\mathbb{Q}$ is finite. Prove that A must be a field. [14]

M6. (i) Let A be an $n \times n$ integer matrix whose entries are all even. Show that the determinant of A is divisible by 2^n. Hence or otherwise, show that if B is an $n \times n$ matrix whose entries are ± 1, then the determinant of B is divisible by 2^{n-1}.

(ii) Let

$$M = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 0 & t & 0 \\ 0 & -4 & 5 & 2 \end{bmatrix}.$$

If $\text{rank}(M) = 2$, calculate t.

[(3+6)+5=14]

M7. For a digraph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the set of vertices and $E = \{e_1, \ldots, e_m\}$ is the set of directed edges, define the incidence matrix $M_{m \times n} = ((m_{ij}))_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ as follows.

$$m_{ij} = \begin{cases} 1 & \text{if } e_i = (v_j, v) \text{ for some } v \in V; \\ -1 & \text{if } e_i = (v, v_j) \text{ for some } v \in V; \\ 0 & \text{otherwise.} \end{cases}$$

If G is strongly connected (i.e., there exists a directed path between any pair of vertices), show that M has $n-1$ linearly independent rows. [14]

M8. Answer **ANY TWO** of the following four questions, each carrying 7 marks.

(a) Show that $p^{q-1} + q^{p-1} \equiv 1 \bmod pq$, for any two distinct primes p and q.

(b) Let p be a prime number and $\mathbb{F}_p$ the field of order p. Consider the polynomial $p(x) = x^2 + x$. Show that there exists an element $c \in \mathbb{F}_p$ such that the polynomial $p(x) + c$ has no root in $\mathbb{F}_p$.

(c) Show that $\displaystyle\int_0^\infty \frac{dx}{x^2 + \sqrt{x}} < \infty$.

(d) Compute the maximum number of edges of a simple graph with n vertices that has exactly k connected components.

[7 + 7 = 14]

Section IV : Physics

Answer any *FIVE* questions.

P1. (i) An ideal gas is enclosed in a vertical cylinder closed by a piston of cross-section α. The atmospheric pressure is constant and is p_0. An external force is applied to lift the piston from a height h_1 to h_2 isothermally. Neglecting the weight of the piston, find the work done by the applied force.

(ii) A system of particles occupying single-particle states and obeying Maxwell-Boltzmann statistics is in equilibrium at an absolute temperature T. The populations having energies 3.22 MeV and 11.5 MeV are 63% and 21%, respectively. Compute the value of T, given that $1\text{eV} = 1.6 \times 10^{-19}$Joule, Boltzmann constant $= 1.38 \times 10^{-23}$ J/K, and $\ln 3 = 1.10$.

[7+7 = 14]

P2. (i) Consider a quantum system whose state is given in terms of an orthonormal set of three vectors $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$ as

$$|\psi\rangle = \frac{\sqrt{3}}{3}|\phi_1\rangle + \frac{2}{3}|\phi_2\rangle + \frac{\sqrt{2}}{3}|\phi_3\rangle.$$

Calculate the probability of finding the system in any one of the states $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$.

Consider now an ensemble of 810 identical systems, each one of them in the state $|\psi\rangle$. If separate measurement is done on each of them, how many systems will be found in each of the states $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$?

(ii) Consider the harmonic oscillator given by

$$H = \frac{p^2}{2} + \frac{x^2}{2},$$

where H is the Hamiltonian, p is the momentum and x is the displacement from the mean position. Let

$$a = \frac{1}{\sqrt{2}}(p - ix),$$

$$a^\dagger = \frac{1}{\sqrt{2}}(p + ix).$$

(a) Show that $[a, a^\dagger] = 1$.

(b) Show that if $|n\rangle$ is an eigenstate of $N = a^\dagger a$ with eigenvalue n, then $a^\dagger|n\rangle$ and $a|n\rangle$ are also eigenstates of N with eigenvalues $(n+1)$ and $(n-1)$, respectively.

[(3+4) + (2+5) = 14]

P3. (i) Consider two elements X and Y. Suppose that X has atomic mass 45, mass density 9000 kgm^{-3}, and forms an fcc lattice. On the other hand, Y has atomic mass 144, mass density 1800 kgm^{-3}, and forms a bcc lattice. Calculate the ratio of their atomic diameters.

(ii) For a free electron gas in three dimensions, show that the density $N(E)$ of states with energy E is proportional to $E^{\frac{1}{2}}$. How does the result change in lower (one and two) dimensions?

[8+6=14]

P4. (i) An *alpha* particle having kinetic energy 8.317 MeV emitted by $^{214}_{84}$Po (also known as RaC′) strikes a $^{14}_{7}$N nucleus at rest to form the compound nucleus $^{18}_{9}$F which immediately emits a proton ($^{1}_{1}$H) and an oxygen ($^{17}_{8}$O) nucleus in mutually opposite directions. The nuclear reaction is expressed as:

$$^{4}_{2}\mathrm{He} \ + \ ^{14}_{7}\mathrm{N} \rightarrow ^{17}_{8}\mathrm{O} \ + \ ^{1}_{1}\mathrm{H} + Q \ (\mathrm{MeV}),$$

where Q is the nuclear reaction energy. Calculate Q and the kinetic energy of a proton. The different nuclear masses involved in the reaction are: ^{4}He = 4.0026 mu; ^{14}N = 14.0031 mu; ^{1}H = 1.0078 mu; ^{17}O = 16.9991 mu. Here, 1mu = $\frac{1}{12}$(mass of a ^{12}C in kg)=1.66×10^{-27} kg has an equivalent energy of 931 MeV.

(ii) What are the kinetics and energetics involved in the following nuclear transformations?

$$^{64}_{29}\mathrm{Cu} \rightarrow ^{64}_{28}\mathrm{Ni},$$

$$^{64}_{29}\mathrm{Cu} \rightarrow ^{64}_{30}\mathrm{Zn}.$$

[9+5=14]

P5. (i) A stream of particles, each of atomic mass 4 and double electronic charge, is accelerated by a potential difference of 100 volts and projected midway between two parallel plates 1 cm apart and 3 cm long. Compute the deflection of the beam on a screen 15 cm beyond the plates, when a potential difference of 20 volts is maintained between the plates.

(ii) An electron and a proton separately move in a plane perpendicular to the magnetic field of the earth with speeds v and $0.002v$ respectively at a place where the magnetic induction has a magnitude B. Calculate the ratio of their radii of trajectories and also the ratio of their periods of revolution.

[8+(3+3)=14]

P6. (i) An impulse J is exerted horizontally on a solid sphere of radius R placed on a frictionless horizontal surface. The line of action of the impulse is at a height h above the center of the sphere. If the rotational and translational kinetic energies of the sphere just after the impulse are equal, then find the value of h in terms of R.

(ii) A spring fixed at one end has natural length l and spring constant k. A mass m is hung on the other end of the spring. Assume that the mass can move only in the vertical direction. Let y be the vertical distance of the mass as measured from the fixed end of the spring. Write the Lagrangian of the system and hence solve the Euler-Lagrangian equation of motion of the mass m.

[6+(3+5) = 14]

P7. (i) Consider the circuit shown in Figure 8. Compute the current through the branch QS in this circuit.

(ii) A sinusoidal voltage of 10 volts rms value at 5 kHz is applied to the series combination of a 10 Ω resistance, $\frac{0.01}{\pi}$ Henry inductance and $\frac{1}{\pi}$ μF capacitance. Calculate the rms current flowing through the circuit. Further, determine the rms voltage across the inductor, capacitor and resistor separately and also the total voltage drop across all these three circuit elements.

[6+(2+3+3)=14]

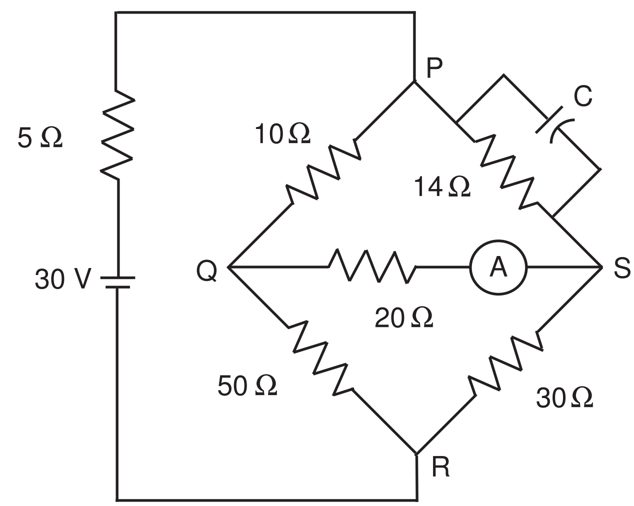

Figure 8: Circuit for Question P7(i)

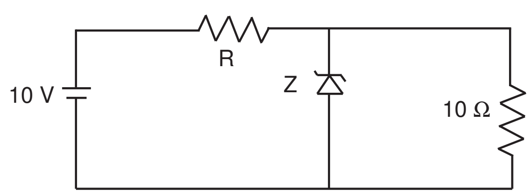

Figure 9: Circuit for Question P8(ii)

P8. (i) Design a combinational circuit which will have three input lines and one output line, so that the output line will be 1 only when exactly two input lines are at 1 state. Use only NOR gates to design the circuit.

(ii) In the circuit shown in Figure 9, Z is a Zener diode with a breakdown voltage of 5 V having maximum power dissipation capacity of 1.5 W. Calculate the minimum and maximum values of R so that the Zener diode operates in its breakdown region.

[7+7=14]

Section III : STATISTICS

Answer any *FIVE* questions.

S1. The probability, p_n, of a family having exactly n children is $\alpha\beta^n$ for $n \geq 1$, with $p_0 = 1 - \alpha\beta(1 + \beta + \beta^2 + \cdots)$, where $\alpha, \beta \in (0, 1)$. Assume that all sex distributions of n children have the same probability. Given that a family has at least one boy, what is the probability that it has two or more boys? [14]

S2. Let X, Y and Z be independent random variables, identically distributed as $\mathcal{N}(0, 1)$.

(a) Find the distribution of

$$\frac{X + Y + Z}{|X - 2Y + Z|}.$$

(b) If X and Y are independent of $W \sim U(-\frac{1}{2}, \frac{1}{2})$, and

$$T = \frac{X + WY}{\sqrt{1 + W^2}},$$

then show that T has a normal distribution.

[7+7=14]

S3. The random variable X has a Poisson distribution with unknown mean λ, where $0 < \lambda < \infty$.

(a) Find the minimum variance unbiased estimator (MVUE) of $e^{-2\lambda}$.

(b) Define an alternative estimator of $e^{-2\lambda}$, namely,

$$S(X) = \begin{cases} 1 & \text{if } X \text{ is even,} \\ 0 & \text{if } X \text{ is odd.} \end{cases}$$

Compare the mean squared error of $S(X)$ with that of the MVUE.

[7+7=14]

S4. (i) Let X be an observation from the probability mass function

$$p(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1,$$

where $0 \leq \theta \leq 1$.

1. Find the maximum likelihood estimator of θ.
2. Find an unbiased estimator of θ.

(ii) Let X_1, X_2 and X_3 be independent and identically distributed Bernoulli(p) random variables, where $0 < p < 1$. Verify whether $X_1 + X_2 + 2X_3$ is sufficient for p.

[(5+3)+6=14]

S5. There are two methods, A and B, of preparing tea. Tom claims that he can identify, 90% of the time, the method of preparation of tea by taking a couple of sips. To test his claim, 5 cups of tea are prepared by method A and 5 cups by method B. These 10 cups are presented in random order to him for tasting. It is decided that Tom's claim will be accepted if he can correctly identify the method of preparation of at least 8 cups. What are the probabilities of errors of Type I and Type II for this test?
[7+7=14]

S6. Consider the following fixed-effects linear model:

$$\begin{aligned} Y_1 &= \beta_1 + \beta_2 + \beta_4 + \epsilon_1, \\ Y_2 &= \beta_1 + \beta_3 + 2\beta_4 + \epsilon_2, \\ Y_3 &= \beta_1 + \beta_2 + \beta_4 + \epsilon_3, \\ Y_4 &= -\beta_2 + \beta_3 + \beta_4 + \epsilon_4, \\ \boldsymbol{\epsilon} &\sim N(\mathbf{0}, \sigma^2 \boldsymbol{I}_4), \end{aligned}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)'$.

(a) Find the full set of error functions.

(b) Hence or otherwise, find an unbiased estimator of σ^2 with maximum possible degrees of freedom.

[8+6=14]

S7. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed $\mathcal{N}(\theta, 1)$ random variables, where it is known that $\theta \geq 0$. Find the maximum likelihood estimator of θ and show that it is better than the sample mean $\overline{X}$ in the sense of having a smaller mean squared error. [7+7=14]

S8. In a block, there are 5 villages having 50, 100, 100, 100 and 50 households. For estimating unbiasedly the total number of children of age less than or equal to 5 years in the block, data are collected according to the following procedure:

(1) 2 villages are selected by *Probability Proportional to Size With Replacement* (PPSWR) sampling.

(2) From each selected village, 5 households are selected by *Simple Random Sampling With Replacement* (SRSWR).

(3) From each selected household, the number of children of age less than equal to 5 years is recorded.

The observations are:

0, 3, 1, 1, 1 from village no. 2,
1, 2, 2, 1, 1 from village no. 5.

What will be the required estimate? Justify your answer.

[14]