2014

BOOKLET No. TEST CODE: **PCB**

Afternoon

TIME: 2 HOURS

GROUP	MAX SCORE
A	**30**
B	**70**

Write your Registration Number, Test Centre, Test Code and the Booklet No. in the appropriate places in the answer-book.

The questions are divided into *two* groups, A and B.

- ANSWER **ALL** QUESTIONS IN **GROUP A**.
- **Group B** consists of the following ***five*** sections:

  I. **Computer Science**
  II. **Engineering and Technology**
  III. **Mathematics**
  IV. **Physics**
  V. **Statistics**

ANSWER QUESTIONS FROM **ONLY ONE** SECTION IN **GROUP B**.

The marks allotted to each question appear within the brackets **[ ]** following it.

YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP! WAIT FOR THE SIGNAL TO START.

Group A

Answer all questions.

A1. Two boxes contain 65 balls of several different sizes and colors. The color of a ball is either white or black or red or yellow. If you take any five balls of the same color, at least two of them will always be of the same size. Prove that there are at least three balls which lie in the same box, having the same color and size. [7]

A2. Let m and n be two integers such that $m \geq n \geq 1$. Count the number of functions $f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, m\}$ of the following two types:

(a) strictly increasing; i.e., whenever $x < y$, $f(x) < f(y)$, and

(b) non-decreasing; i.e., whenever $x < y$, $f(x) \leq f(y)$.

[3 + 5 = 8]

A3. Prove that $2^n + 1$ can never be a multiple of 7 for any positive integer n. [5]

A4. For any two points $p_1 = (a_1, b_1)$ and $p_2 = (a_2, b_2)$ in $\mathbb{R}^2$, such that $a_1 \neq a_2$ and $b_1 \neq b_2$, we define

$$f(p_1, p_2) = \begin{cases} LU, & \text{if } a_1 < a_2 \text{ and } b_1 > b_2, \\ RU, & \text{if } a_1 > a_2 \text{ and } b_1 > b_2, \\ LB, & \text{if } a_1 < a_2 \text{ and } b_1 < b_2, \\ RB, & \text{if } a_1 > a_2 \text{ and } b_1 < b_2. \end{cases}$$

Suppose $q_1 = (x_1, y_1), q_2 = (x_2, y_2), \cdots, q_k = (x_k, y_k)$ are k given points in $\mathbb{R}^2$ such that for $1 \leq i < j \leq k$, we have $x_i \neq x_j$ as well as $y_i \neq y_j$. Let

$$P = \{(a, b) \in \mathbb{R}^2 \ : \ a \notin \{x_1, x_2, \cdots, x_k\}, b \notin \{y_1, y_2, \cdots, y_k\}\}.$$

The position of $p \in P$ with respect to $q_1, q_2, \cdots, q_k$ can be represented by a vector $(f(p, q_1), f(p, q_2), \cdots, f(p, q_k))$. How many such distinct vectors are possible? [10]

Group B

Section I : Computer Science

Answer any *FIVE* questions.

C1. (a) Read the C code given below. What would be the output of the following program? Justify your answer.

```
#include <stdio.h>

int myrecurse(int a, int b){
  return (b == 1 ? a: myrecurse(a, b-1) + a);
}

main() {
  int a[]= {2,3,4,5,6};
  int i, x;

  for (i = sizeof(a)/sizeof(*a), x = 0; --i; )
    x += myrecurse(*(a+i), a[i-1]);

  printf("\nThe result is: %d", x);
}
```

(b) Suppose you are given a sequence $S =< x_1, x_2, \ldots, x_n >$ of n integers. For $1 \leq i \leq j \leq n$, the sequence $< x_i, x_{i+1}, \ldots, x_j >$ of consecutive elements is called a *subsequence* of S. The sum of a subsequence is the sum of the integers in that subsequence. Give an $O(n \log n)$ algorithm to determine whether the given sequence S has a subsequence whose sum is zero, and justify the correctness of the algorithm.

[6 + (4 + 4) = 14]

C2. $S[1 \ldots n]$ is a sorted array of n integers, where n is even. A new array S' is generated by swapping some elements in odd-numbered positions in the first half of S with some elements in odd-numbered positions in the second half of S. Note that the elements in the even-numbered positions are the same in both S and S', whereas each element in an odd-numbered position in S takes part in at most one swap. Write an $O(\log n)$ algorithm that takes S' and an integer x as inputs and finds whether x is present in S' or not. [14]

C3. (a) Prove that the language $\{a^N : N \text{ is a composite number}\}$ is not regular.

(b) Write a suitable grammar to generate the following language: $\{a^n b^{n+m} a^m : m, n \text{ are integers} \geq 1\}$.

[8 + 6 = 14]

C4. (a) Consider the following grammar over the alphabet $\{a, b, c\}$, with S as the start symbol.

$$
\begin{array}{lcc|c}
S & \rightarrow & bAb & bBa \\
A & \rightarrow & aS & CB \\
B & \rightarrow & b & BC \\
C & \rightarrow & c & cC
\end{array}
$$

Show that the grammar is ambiguous.

(b) Suppose that in a hierarchical memory subsystem with cache and main memory (MM), the MM cycle time is 480 ns, cache cycle time is 64 ns, probability of cache hit is 0.8 for read and 0.9 for write, probability of write is 0.3 and probability of a page being dirty is 0.2. Compute the average data access time for *write-back* scheme.

[7 + 7 = 14]

C5. Suppose we have a relation Student(name, department, marks). For each of the following queries, write the query in relational algebra and explain informally why your answer works. You may use only the following operators of relational algebra: union, intersection, difference, selection, projection, products, joins (theta and natural), renaming.

(a) List the department(s).

(b) Find department(s) with at least two students of different names.

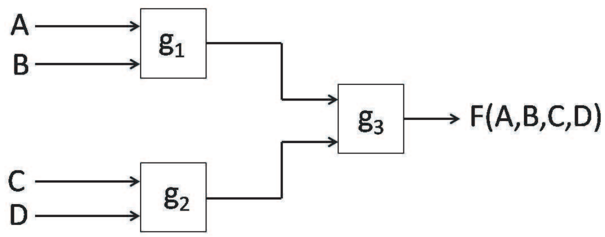
(c) Find student name(s) with the lowest marks.

[3+6+5 = 14]

C6. (a) Consider the combinational circuit below where g_1, g_2 and g_3 are three 2-input, unknown logic gates (such as AND, OR, NAND, NOR, XOR, XNOR). The output Boolean function realized by the circuit is

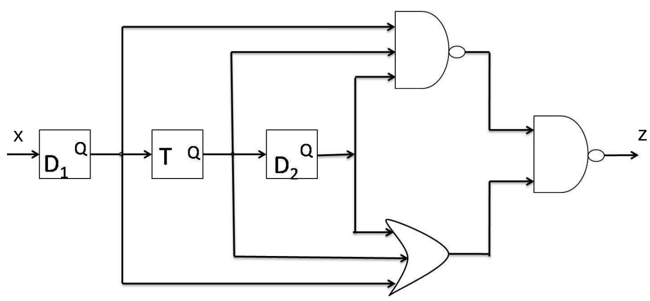
$$F(A, B, C, D) = ((AB\bar{C}) \oplus (\bar{A}CD)) + ((AB\bar{D}) \oplus (\bar{B}CD))$$

where $\oplus$ denotes the XOR operation.



Determine the gate types of g_1, g_2 and g_3.

(b) Consider the synchronous sequential circuit below, which has one input x, one output z and two D-flipflops D_1, D_2 and one T-flipflop T. All flipflops are clocked with the same clock.



The circuit receives the following binary sequence X at the input x, where X is fed to the circuit starting from the most significant bit (MSB) end:

X : 10011100
time (t) $\rightarrow$

Assuming that the initial state of the machine is D_1TD_2 = {010}, determine the output sequence that will appear at z.

[8 + 6 = 14]

C7. (a) Consider sending a packet of F bits over a path of Q links. Each link transmits at R bps. The network is lightly loaded so that there are no queuing delays. Propagation delay is also negligible.

i. Suppose the network is a packet-switched datagram network and a connection-oriented service is used. Suppose each packet has $(h \times F)$ bits of header where $0 < h < 1$. Assuming t_s setup time, how long does it take to send the packet?

ii. Suppose that the network is a circuit-switched network. Furthermore, suppose that the transmission rate of the circuit between source and destination is $\frac{R}{24}$ bps. Assuming t_s setup time and no bits of header appended to the packet, how long does it take to send the packet?

(b) A system has four processes P1 through P4 and two resource types R1 and R2. It has 2 units of R1 and 3 units of R2. Given that:

- P1 requests 2 units of R2 and 1 unit of R1.
- P2 holds 2 units of R1 and 1 unit of R2.
- P3 holds 1 unit of R2.
- P4 requests 1 unit of R1.

i. Show the resource graph for this state of the system.

ii. Is the system in deadlock? If so, which processes are involved? If not, give a sequence in which the process will be executed.

[(4+4) + 6 = 14]

Section II : Engineering and Technology

Answer any *FIVE* questions.

E1. A 20 kg uniform cylindrical roller of radius 250 mm, initially at rest, is pulled horizontally by a force of 90 N as shown in Figure 1. Assume that it rolls without slipping.

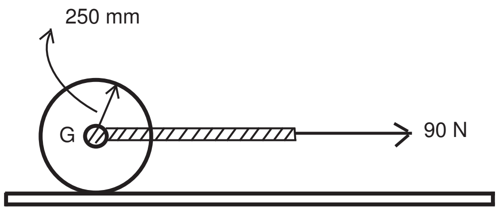


Figure 1

Determine

(a) the velocity of the centre G after it has moved through a distance of 1.5 m, and

(b) the minimum frictional force required to prevent slipping.

[6 + 8 = 14]

E2. (a) 5 kg of steam contained within a piston-cylinder assembly (as shown in Figure 2) undergoes an expansion from State 1 to State 2. The specific internal energy at State 1 is 2709.9 kJ/kg while at State 2 it is 2659.6 kJ/kg. During the process there is heat transfer of 80 kJ to the steam. The paddle wheel performs 18.5 kJ of work on the steam. There is no significant change in the kinetic or potential energy of the steam. Determine the work done by the steam on the piston.

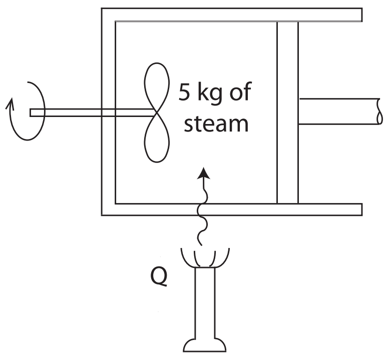


Figure 2

(b) A solid cylinder of length 50 cm is placed on its base upon a horizontal surface and subjected to a vertical compressive force of 1000 N, directed downwards and distributed uniformly over the end face. What will be the resulting change in volume of the cylinder? Assume the Poisson's ratio of the material of the cylinder to be 0.3 and Young's Modulus to be 2×10^{11} N/m^2.

[5 + 9 = 14]

E3. (a) An arrangement of six resistors, each of value r, is connected to a battery B of emf 10V and internal resistance 1Ω, as shown in Figure 3. The voltmeter V reads 9V when the switch K is closed. Find the value of r.

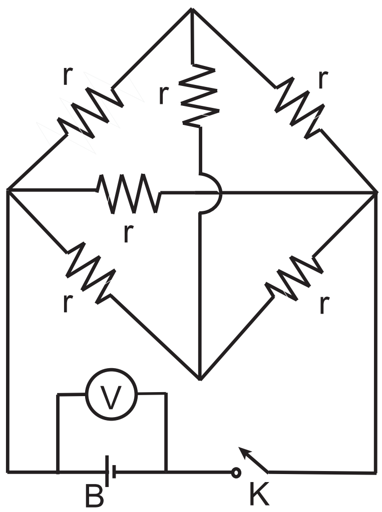


Figure 3

(b) You are required to construct a parallel plate air gap capacitor with minimum volume that can store 80 mJ of energy. Assume that the dielectric strength of air is 3×10^6 volt/metre and the permittivity of free space, $\epsilon_0 = \frac{10^{-9}}{36\pi}$ farad/metre. Suppose that you now fill the space between the two parallel plates of this capacitor with a dielectric material whose dielectric strength is 3×10^8 volt/metre with a dielectric constant of 5. What fraction of the original volume of air must this dielectric have, for it to be still able to store 80 mJ of energy?

[7 + 7 = 14]

E4. The switch S in Figure 4 is thrown to position ‘a’ at $t = 0$. The switch is thrown to position ‘b’ at $t = 1$ sec.

(i) Sketch the waveform of the current $i(t)$ through R_2 for $t \geq 1$ sec.

(ii) Determine the maximum value of $i(t)$.

(iii) Find the time at which $i(t)$ reaches its maximum value.

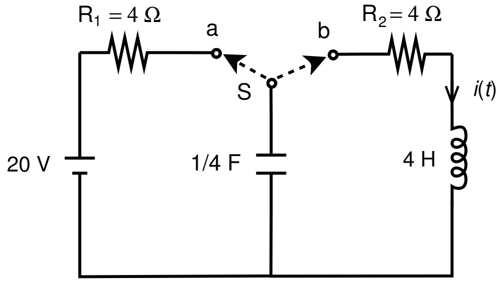


Figure 4

[5 + 5 + 4 = 14]

E5. (a) Calculate the inductance of a toroid of length 2 cm, internal radius 3 cm and external radius 6 cm, having 1000 turns of wire. Assume that the core has a magnetic material with relative permeability of 5.

(b) Consider the infinite ladder circuit in Figure 5 where the L and C values are very small. For example, assume that L is in the range of 0.1 pH to 1 pH and C is in the range of 0.1 pF to 1 pF. Derive an expression for the value of the input impedance of this circuit as seen from the terminals $1, 1'$. Assume that the operating frequency is less than 1 GHz.

[7 + 7 = 14]

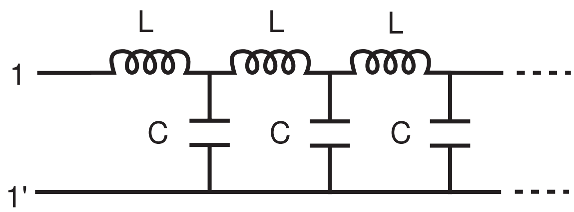


Figure 5

E6. The following code in C language converts an unsigned fractional decimal number to binary by considering the integral and fractional parts separately. The binary representation of the fractional part is truncated after 4 decimal points. For example, for an input of 37.25 the output would be 100101.0100 You are required to fill in the blanks appropriately in the program.

```
int convert(double fraDecimal){
double fraBinary, bFractional=0.0,
                dFractional,fraFactor=0.1;
long int dIntegral, bIntegral=0,
                remainder,intFactor=1;
int i;
dIntegral=fraDecimal;
dFractional= _______________ - dIntegral;
while(dintegral!=0){
    remainder=dIntegral%_______________;
    bIntegral+= remainder * _______________;
    dIntegral = dIntegral/ _______________;
    intFactor= intFactor * _______________;
}
for(i=0; i<4; i++) {
    dFractional+= _______________;
    temp= dFractional;
    bFractional+= fraFactor * temp;
    if(temp==1) dFractional-= temp;
    fraFactor= fraFactor/_______________;
}
fraBinary= bIntegral+bFractional;
printf("Equivalent Binary value: %16.4f",fraBinary);
}
```

[14]

E7. Consider the small signal amplifier as shown in Figure 7 where $R_e = 100\Omega$, $R_s = 10\Omega$ and $R_L = 1.01k\Omega$.

(a) Assuming that $h_{fe} = 100$ and $h_{ie} = 90\Omega$ and neglecting h_{re} and h_{oe}, (where h_{fe}, h_{ie}, h_{re} and h_{oe} are the usual h-parameters in common-emitter mode), show that the voltage gain $A_v = V_o/V_s$ is approximately equal to 10.

(b) If the relative change $\frac{dA_v}{A_v}$ must not exceed a specified value ψ due to variations in h_{fe}, then show that the required value of R_e is given by

$$R_e \geq \frac{dh_{fe}/h_{fe}}{\psi} - 1$$

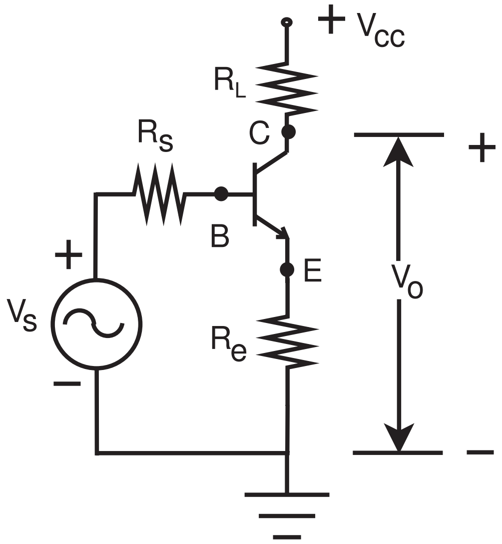


Figure 7

[7 + 7 = 14]

E8. Consider the following circuit in Figure 8 with two OP-AMPs A_1, A_2, two diodes D_1, D_2, three resistors and one capacitor C. Assume that all elements are ideal.

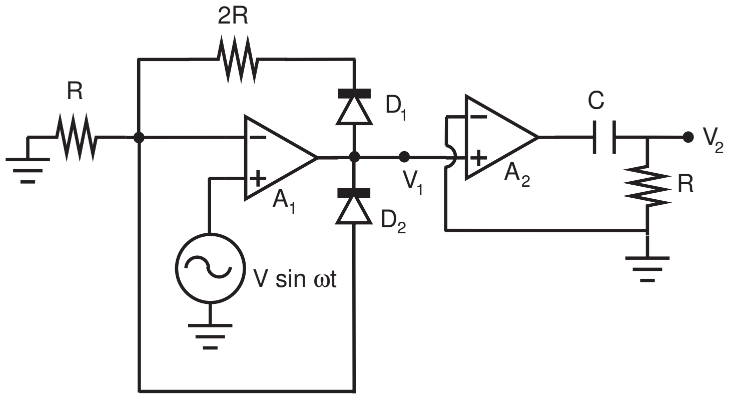


Figure 8

The circuit receives an input signal $V \sin \omega t$ as shown. Assume that RC is very small compared to the period of the input signal.

(a) Derive an expression for the maximum voltage observed at node V_1.

(b) Sketch the waveforms at nodes V_1 and V_2 as function of time.

[(4 + (5 + 5) = 14]

E9. (a) A long shunt compound wound DC generator supplies 48A load current at 220V. If the series and parallel resistances are 0.4Ω and 110Ω, what is the efficiency of the generator, given that the armature resistance is 0.6Ω, loss at each brush is equivalent to 1Ω and iron losses are neglected?

(b) A 2000/200V, 40 kVA two-winding transformer is connected as an auto-transformer with additive polarity. Suppose that the transformer supplies to a load with power factor 1. Calculate

(i) the percentage increase in kVA rating, and

(ii) the power transferred from input to output inductively.

[7 + 7 = 14]

E10. (a) Consider the combinational circuit of Figure 9(a) where g_1, g_2 and g_3 are three 2-input, unknown logic gates (such as AND, OR, NAND, NOR, XOR, XNOR). The output Boolean function realized by the circuit is

$$F(A,B,C,D) = ((AB\bar{C}) \oplus (\bar{A}CD)) + ((AB\bar{D}) \oplus (\bar{B}CD))$$

where $\oplus$ denotes the XOR operation. Determine the gate types of g_1, g_2 and g_3.

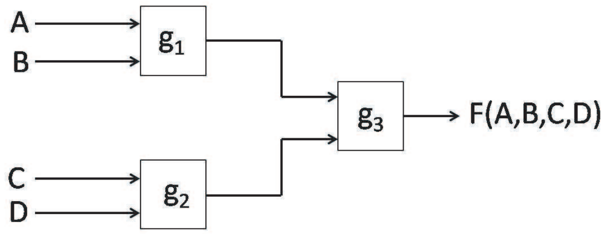


Figure 9(a)

(b) Consider the synchronous sequential circuit of Figure 9(b), which has one input x, one output z and two D-flipflops D_1, D_2 and one T-flipflop T. All flipflops are clocked with the same clock. The circuit receives the following binary sequence X at the input x, where X is fed to the circuit starting from the most significant bit (MSB) end:

X : 10011100
time (t) $\rightarrow$

Assuming that the initial state of the machine is $D_1TD_2 = \{010\}$, determine the output sequence that will appear at z.

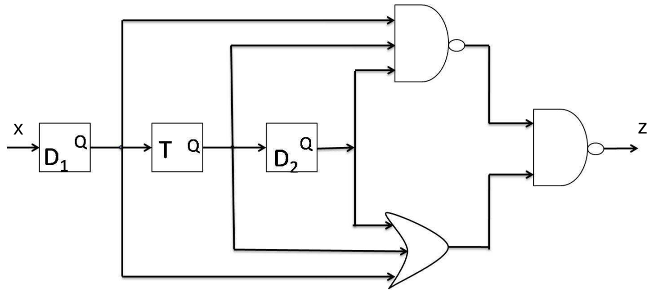


Figure 9(b)

[8 + 6 = 14]

Section III : Mathematics

Answer any *FIVE* questions.

M1. (a) For real $x > 0$, define

$$u_n(x) = \frac{1}{x+1} + \frac{2x}{(x+1)(x+2)} + \frac{3x^2}{(x+1)(x+2)(x+3)} + \cdots + \frac{nx^{n-1}}{(x+1)(x+2)\cdots(x+n)}.$$

Find $\lim_{n\to\infty} u_n(x)$.

(b) Find the sum of the infinite series

$$\tfrac{1}{2}x^2 + \tfrac{1}{3}\cdot\tfrac{1}{2}x^3 + \tfrac{1}{4}\cdot\tfrac{1}{3}x^4 + \cdots,$$

where $0 < x < 1$.

[7+7=14]

M2. Give an example of a continuous onto function $f : (0,2) \to (0,2)$ and a countably infinite subset A of $(0,2)$ such that f is not differentiable at every point of A and differentiable at every point of $(0,2) \setminus A$?

[14]

M3. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 3e^{4x}.$$

(b) Let R be the region in the xy-plane bounded by the following curves:

$$y = e^x, y = \frac{1}{2}e^x, y = \frac{\pi}{2}, y = \pi.$$

Evaluate the double integral

$$\iint_R e^x \frac{\sin y}{y}\, dxdy.$$

[7+7=14]

M4. Consider the 2×2 real matrix $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a, b, c, d > 0$ and $a + b = c + d = 1$. Let λ_n and μ_n be the eigenvalues of P^n.

(a) Show that $\lambda_n \neq \mu_n$.

(b) Assume that $\lambda_n > \mu_n$. Show that $\mu_n \to 0$ as $n \to \infty$.

(c) Assume that P is a symmetric matrix. Show that

$$P^n \to \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ as } n \to \infty.$$

[3+4+7=14]

M5. Let $\mathcal{M}$ be the set of all 2×2 real matrices. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}$, define

$$\|A\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

(a) Show that $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in \mathcal{M}$.

(b) Show that the series

$$I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

converges for every $A \in \mathcal{M}$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

[5+9=14]

M6. Let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let M denote the column space of X and N be the orthogonal complement of M in $\mathbb{R}^5$.

(a) Find an orthonormal basis of N.

(b) Find the vector $a \in N$ such that the Euclidean distance from a to v is the smallest among all vectors in N.

[4+10=14]

M7. (a) Let G be an abelian group. Suppose that H and K are subgroups of G such that $H \cap K = \{e\}$ and $HK = G$. Show that G is isomorphic to $H \times K$.

(b) Let R denote the ring of integers and U be the ideal of R consisting of all multiples of 19. If V is an ideal of R such that $U \subset V \subset R$, then prove that either $V = U$ or $V = R$.

[7+7=14]

M8. (a) A directed graph without self loops is called a *tournament* if between any two vertices u and v exactly one of the edges (u, v) or (v, u) is present in the graph.

Show that in any tournament, there is a vertex from which any other vertex can be reached by travelling along at most two directed edges.

(b) Show that from a simple connected undirected graph on n vertices and having m edges, one can always pick $m - n + 1$ edges in such a way that at least one edge is picked from every cycle in the graph.

[8+6=14]

Section IV : Physics

Answer any *FIVE* questions.

P1. (a) A bead slides on a wire in the shape of a cycloid described by equations

$$x = a(\theta - \sin\theta), \qquad y = a(1 + \cos\theta),$$

where $0 \leq \theta \leq 2\pi$.
Find (i) the Lagrangian function, and (ii) the equation of motion.

(b) Consider the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 - w^2x^2)e^{\gamma t}$$

for the motion of a particle of mass m in one dimension (x). The constants m, γ and w are real and positive.

(i) Find the canonical momentum and from this construct the Hamiltonian function.

(ii) Is the Hamiltonian an invariant for the motion? Explain.

[(3 + 4) + (5 + 2) = 14]

P2. An electron is at rest in an oscillating magnetic field

$$\mathbf{B} = B_0 \cos(\omega t)\hat{z},$$

where B_0 and ω are constants.

(a) Construct the Hamiltonian matrix for this system.

(b) Let $\chi(t)$ denote the state of the electron at time t. The electron starts out (at $t = 0$) in the spin-up state with respect to the x-axis (i.e., $\chi(0) = \chi_+^{(x)}$). Derive an expression of $\chi(t)$.

(c) Find the probability of getting $S_x = -\hbar/2$, where S_x corresponds to the x-component of spin angular momentum.

(d) What is the minimum value of the amplitude B_0 to force a complete flip in S_x?

[2 + 7 + 3 + 2 = 14]

P3. (a) The electric potential at a point in a specific region is given by the expression

$$V(r) = A\frac{e^{-\lambda r}}{r},$$

where A and λ are constants and $\mathbf{r}$ is the position vector (with respect to a reference point as origin). Find the electric field $\mathbf{E}(\mathbf{r})$, the charge density $\rho(\mathbf{r})$, and the total charge Q.

(b) Find the magnetic field at the center of a regular n-sided polygon, carrying a steady current I. Let R be the perpendicular distance from the center to any side. What is the value of this magnetic field when $n \to \infty$?

[(2 + 5 + 2) + (3 + 2) = 14]

P4. (a) Lattice constant of a cubic lattice is a. Calculate the spacing between (2 1 1), (1 1 0), (1 1 1) and (1 0 1) planes.

(b) Show that for a crystal of cubic symmetry the direction [h, k, l] is perpendicular to the plane (h k l).

(c) The diamond crystal structure has the cube edge of 356Å. Calculate the distance between the nearest-neighbor atoms.

[6 + 5 + 3 = 14]

P5. (a) What would be the electron velocity at the first Brillouin zone edge of a one dimensional crystal of atomic spacing 5Å? Assume that the ratio of Planck's constant to mass of electron is equal to 7.3×10^{-4} J.s/kg.

(b) In a one dimensional crystal with atomic spacing 2.5Å, calculate the free electron energy at which first Bragg reflection occurs.

(c) The collection of electrons in a metallic solid may be modeled as a three-dimensional free electron gas. Find the allowed values of wave-vector k, and sketch the appropriate Fermi sphere in k-space (use periodic boundary conditions with length L).

[4 + 4 + (3 + 3) = 14]

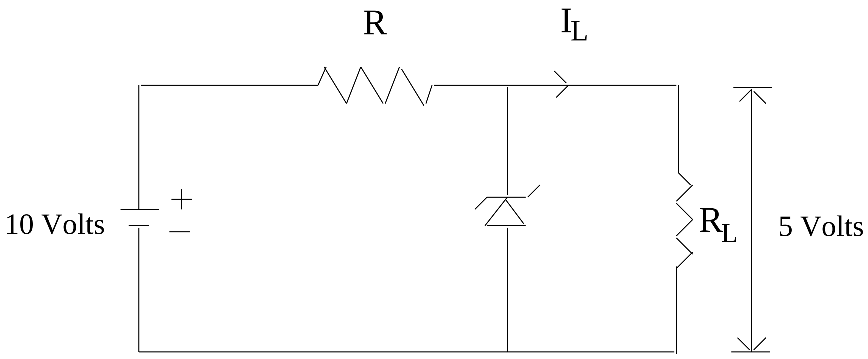
P6. Find the magnetic moment, in unit of Bohr magneton, of an atom in the state $^3p_2$. In how many sub-states will the state split, if the atom is put in a weak magnetic field? Show the splitting diagram.

[6 + 3 + 5 = 14]

P7. The zero-voltage barrier height of an abrupt p-n junction is 0.35 volt. Assume that the concentration N_A of acceptor atoms on the p-side is much smaller than that of donor atoms on the n-side, and $N_A = \frac{10^{19}}{\pi} \mathrm{m}^{-3}$. Calculate the width of the depletion layer for an applied reverse voltage of 14.65 volts.

If the cross-sectional area of the diode is 1 sq. mm, calculate the space charge capacitance corresponding to this applied reverse voltage. (Assume that the permittivity of the material is $12 \times 10^{-9}/36\pi$ farad/metre). [8 + 6 = 14]

P8. Consider the following circuit which is designed to derive a 5 volts DC supply using a Zener diode from a 10 volts regulated DC supply. The load current I_L through R_L can, however, vary from 0 to 1 amp.



What should be the required characteristic of the Zener diode to serve this purpose and what should be the appropriate value of the resistance R? Justify your answers. [8 + 6 = 14]

Section V : Statistics

Answer any *FIVE* questions.

S1. Let A_1, A_2, A_3 be independent events of probability $\frac{1}{2}$ each. Let $B_{ij} = (A_i \Delta A_j)^c$, $1 \le i < j \le 3$. Here $A_i \Delta A_j = (A_i \cap A_j^c) \cup (A_i^c \cap A_j)$. Are B_{12}, B_{13} and B_{23} independent ? Justify your answer. [14]

S2. Suppose that (X, Y) has a bivariate normal distribution with $E(X) = E(Y) = 0$ and $\text{Var}(X) = \text{Var}(Y) = 1$. Let $\text{Cov}(X, Y) = 0.5$. Find the value of $P(X > 0 \text{ and } Y > 0)$ with justification. [14]

S3. Let us consider the hypotheses H_0 : X has p.d.f. $\frac{1}{4}e^{-\frac{1}{2}|x|}$ and H_A : X has p.d.f. $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, where $-\infty < x < \infty$. Consider the test that rejects H_0 if and only if $|X| < 1$. Let α denote the size of this test.

(a) Find the value of α.

(b) Show that the above test is *most powerful* size α test and find its power.

[4+10 = 14]

S4. Let X be a discrete random variable taking values 1 through 6. The probability distribution of X depends on an unknown parameter θ. The following table gives the probability distributions of X under $\theta = \theta_0$ and $\theta = \theta_1$.

X	1	2	3	4	5	6
$\theta = \theta_1$	0.15	0.15	0.15	0.15	0.15	0.25
$\theta = \theta_0$	0.15	0.20	0.15	0.10	0.10	0.30

Consider the problem of testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, based on a single observation X.

(a) Find a *most powerful test* of level 0.3.

(b) Show that there are infinitely many *most powerful tests* of level 0.3 for this testing problem.

[7+7=14]

S5. Let $\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n}$ be iid Normal$(\boldsymbol{\mu}_{p\times 1}, \boldsymbol{\Sigma}_{p\times p})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown, $n < p$ and $p \geq 3$.

(a) Suggest an unbiased estimator $\hat{\boldsymbol{\Sigma}}$ for $\boldsymbol{\Sigma}$.

(b) Is $\hat{\boldsymbol{\Sigma}}$ positive definite ? Justify your answer.

(c) Is it possible to obtain the maximum likelihood estimator of Σ ? Justify your answer.

[4+5+5=14]

S6. (a) Let $X_1, \ldots, X_n$ be iid Normal$(\mu, 1)$, where $\mu \in (-\infty, \infty)$ is unknown. Find the Uniformly Minimum Variance Unbiased Estimator of e^{μ}.

(b) Let $X_1, \ldots, X_n$ be iid Normal(μ, σ^2) where $\mu \in (-\infty, \infty)$ and $\sigma^2 > 0$ are both unknown. Find the Uniformly Minimum Variance Unbiased Estimator of μ^3.

[7+7=14]

S7. Let $X_1, X_2, \ldots, X_d, \ldots$ be such that for every $d \geq 2$, $(X_1, \ldots, X_d)$ has a multivariate normal distribution with mean $(\mu, \ldots, \mu)$ and a covariance matrix $((\sigma_{i,j}))_{d\times d}$ such that $\sigma_{i,i} = \sigma^2$ for $i = 1 \ldots, d$ and $\sigma_{i,j} = \sigma^2 \rho$ for all $1 \leq i \neq j \leq d$.

(a) Show that $\rho \geq 0$.

(b) Find the regression equation of X_{d+1} on $X_1, \ldots, X_d$.

(c) Find the multiple correlation of X_{d+1} with $(X_1, \ldots, X_d)$.

[3+7+4=14]

S8. Consider a finite population of N distinct units. Let Y_i be the value of the variable of interest corresponding to the i-th unit $(i = 1, 2, \ldots, N)$. Let s be a sample of size n drawn from the population by simple random sampling with replacement.

(a) Show that $\frac{1}{1-(1-\frac{1}{N})^n} \sum_{i\in s} Y_i$ is an unbiased estimate of the population total $\sum_{i=1}^{N} Y_i$.

(b) Derive an expression for the variance of this estimate.

[6+8=14]