2009

BOOKLET No. TEST CODE: **CS**

Afternoon

TIME: 2 HOURS

GROUP	MAX SCORE
A	**30**
B	**70**

Write your Name, Registration Number, Test Center, Test Code and the Booklet No. in the appropriate places on the answer-booklet.

The questions are divided into *two* groups, A and B. Group B consists of *five* sections. ANSWER ALL QUESTIONS IN GROUP A AND FIVE QUESTIONS FROM ONLY ONE OF THE FIVE SECTIONS IN GROUP B. The figures inside the square brackets [ ] following each question denote the marks allotted to it.

IN ANSWERING QUESTIONS, ALL NECESSARY STEPS SHOULD BE CLEARLY SHOWN. WRITE NEATLY AND INDICATE IN A SEPARATE LINE THE FINAL ANSWER OF EACH QUESTION. ALL ROUGH WORK MUST BE SCRATCHED THROUGH.

YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP! WAIT FOR THE SIGNAL TO START.

Group A

1. Let k be a fixed positive integer greater than or equal to 2. Let

$$\begin{aligned} f_1(x) &= \tfrac{x+k-1}{k} \text{ for } x \in \mathbb{R}, \text{ and} \\ f_n(x) &= f_1(f_{n-1}(x)) \text{ for } x \in \mathbb{R} \text{ and } n \geq 2. \end{aligned}$$

   Find $\lim_{n\to\infty} f_n(x)$, if it exists, for $x \in \mathbb{R}$. [10]

2. Let $a < b < c < d$ be four real numbers, such that all six pairwise sums are distinct. The values of the smallest four pairwise sums are 1, 2, 3 and 4 respectively. What are the possible values of d? Justify your answer. [10]

3. Consider the 5×10 matrix A as given below.

$$A = \begin{bmatrix} 1 & 6 & 11 & 16 & 21 & 26 & 31 & 36 & 41 & 46 \\ 2 & 7 & 12 & 17 & 22 & 27 & 32 & 37 & 42 & 47 \\ 3 & 8 & 13 & 18 & 23 & 28 & 33 & 38 & 43 & 48 \\ 4 & 9 & 14 & 19 & 24 & 29 & 34 & 39 & 44 & 49 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 \end{bmatrix}$$

   Let a set of ten distinct elements $b_1, b_2, \ldots, b_{10}$ be chosen from A such that exactly two elements are chosen from each row and exactly one from each column. Show that $b_1 + b_2 + \cdots + b_{10}$ is always equal to 255.

   [10]

Group B

Section (i): Mathematics

1. Define a sequence $\{x_n\}_{n=1}^{\infty}$ as follows:

$$\begin{aligned} x_1 &= 0, \\ x_{n+1} &= a x_n + \tfrac{1}{n}, \quad \text{for } n \geq 1 \text{ and some } a > 0. \end{aligned}$$

   Show that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded iff $0 < a < 1$. [14]

2. (a) Test the following series for convergence

$$\sum_{n=1}^{\infty} (n^{\frac{1}{n}} - 1)^n$$

   (b) Find all the limit points of the set

$$A = \{n^2 + \frac{1}{m^2} : n, m \in Z, m \neq 0\},$$

   where Z is the set of integers.

   [7+7=14]

3. Let n be the cardinality of a set A and m the cardinality of each of the sets B and C, where $B, C \subseteq A$.

   (a) What are the possible values of $|B \bigcup C|$, where $|S|$ denotes the cardinality of the set S?

   (b) For a given value of k, in how many ways can the elements of B and C be chosen so that $|B \bigcup C| = k$?

   [5+9=14]

4. (a) Let f be a real-valued function such that

$$|f(x) - f(y)| \geq |x - y|, \forall x, y \in R.$$

   Show that there do not exist real numbers a, b, c such that

$$a < b < c \text{ with } \mathrm{f(a)} < \mathrm{f(c)} < \mathrm{f(b)}.$$

(b) Find $\lim_{x\to 0} \frac{1}{x} \int_0^{x^2} f(t)\, dt$, where

$$f(t) = \begin{cases} \frac{\sin t}{t}, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases}$$

[7+7=14]

5. Consider the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

(a) Find one eigen-value of this matrix and its corresponding eigen-vector. Justify your answer.

(b) Show that the sum of three of its eigen-values is 1.

[7+7=14]

6. (a) Let A be a square matrix such that $A^2 = mA$, where m $(\neq 0)$ is any real number. Show that

$$\text{tr}(A) = m \times \text{rank}(A),$$

where $\text{tr}(A)$ denotes the trace of A.

(b) Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & \sqrt{3} \\ 0 & \sqrt{3} & 6 \end{pmatrix}$$

Find $\text{tr}(A^{-1})$.

[9+5=14]

7. Consider the two sets given below and the operations defined on them.

- $S_1 = \{1, 2, \ldots, 20\}$ with the operation '*' as multiplication modulo 21.
- $S_2 = \{1, 2, \ldots, 22\}$ with the operation '.' as multiplication modulo 23.

(a) Are $(S_1, *)$ and $(S_2, .)$ groups? Justify your answer using the definition of a group.

(b) In case any one of them is a group, find the inverse of 7 in it and also find a non-trivial subgroup of that group.

[7+7=14]

8. (a) Find the surface satisfying the differential equation

$$x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = 0,$$

with boundary conditions $u = 0$ when $y^2 = 4ax$, and $u = 1$ when $y^2 = -4ax$.

(b) Find real numbers α and β such that the differential equation

$$\frac{d^2y}{dx^2} + (\alpha - 3)\frac{dy}{dx} + \alpha y = 0$$

has $y = A$ cos$\beta x + B$ sinβx as its general solution for arbitrary constants A, B.

[7+7=14]

Section (ii): Statistics

1. (a) Let (X, Y) be a random vector having a joint bivariate distribution with density

$$f(x, y) = \begin{cases} 1 & \text{for } 0 < y < x - \lfloor x \rfloor \text{ and } 0 < x < 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ refers to the largest integer *strictly less than* x. Find $E(X)$, $E(Y)$, Var(X) and Var(Y).

(b) Let X be a random variable with density

$$f(x) = \begin{cases} x & \text{for } 0 < x \leq 1 \\ x - 1 & \text{for } 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of Y where $Y = |X - 1|$. [7+7=14]

2. (a) Does there exist a pair of random variables (X, Y) satisfying

   (i) Var(X) = Var(Y) = 1,

   (ii) Var($X + Y$) = 2 and

   (iii) Var($X - Y$) = $\frac{1}{2}$?

   If your answer is yes, give an example of such a pair of random variables. Otherwise, justify your assertion.

   (b) Suppose that a sequence of random variables $\{X_0, X_1, X_2, \dots\}$ is defined by the following recurrence relation

$$X_0 = 0, \ X_n = pX_{n-1} + Z_n, \ \text{for } n = 1, 2, \dots,$$

   where $0 < p < 1$ is fixed and $Z_1, Z_2, Z_3, \dots$ are random variables satisfying Var(Z_n) =1 for $n \geq 1$ and Cov(Z_n, Z_m) = $\frac{1}{3}$ for $n, m \geq 1$ and $n \neq m$. Find Var(X_3).

[5+9=14]

3. A binary message, either 0 or 1, is to be sent from location A to location B. It is decided to transmit the value +2 if the message is 1 and value -2 if the message is 0. There is a random transmission error y so that a value x sent from A is received as $R = x + y$ at B. Suppose the decoding at B follows the rule that if $R \geq 0.05$, the message is decoded as 1 while it is decoded as 0 if $R < 0.05$. If we assume that $y \sim N(0, 1)$, which of the two errors is more likely, decoding a message 1 as 0 or a message 0 as 1 ? [14]

4. Let $X_1, \dots, X_n$ be iid observations from a $N(\mu, 1)$ distribution, where μ is an unknown real number.

   (a) Find the Minimum Variance Unbiased Estimator of e^{μ} based on $X_1, \dots, X_n$.

   (b) Show that this estimator does not attain the Cramer-Rao lower bound for the variance of any unbiased estimator of e^{μ} based on n iid observations from the assumed distribution.

[7+7=14]

5. (a) Let $X_1, \ldots, X_n$ be iid $U(\theta - 1, \theta + 1)$ where θ is an unknown real number. Show that for any real number $\alpha \in (0, 1)$,

$$\alpha(X_{(n)} - 1) + (1 - \alpha)(X_{(1)} + 1)$$

is a maximum likelihood estimator for the unknown θ, where $X_{(1)}$ and $X_{(n)}$ are the smallest and largest sample observations respectively.

(b) Let $X_1, \ldots, X_n$ be iid $N(\mu, 1)$ where μ is only known to belong to the set of all integers. Find a maximum likelihood estimator for μ based on $X_1, \ldots, X_n$.

[7+7=14]

6. (a) Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, 1)$. We would like to test

$$H_0 : \mu = 0 \text{ vs. } H_1 : \mu = 1,$$

with $\alpha = 0.05$ and $\beta \leq 0.05$, where α and β are the probabilities of type I error and type II error respectively. Is it possible to find such a test for $n = 8$? Is it possible for $n = 100$?
You may take $\Phi(1.64)$ as 0.95, where $\Phi(.)$ is the cumulative distribution function of the standard normal variate.

(b) Suppose we have a single random observation from a discrete probability distribution giving positive probabilities to only the points 1, 2, 3 and 4. It is known that the probability distribution is dependent on an unknown parameter θ which can take either the value 10 or the value 20. If $\theta = 10$, the distribution assigns probabilities 0.15, 0.25, 0.30 and 0.30 to the values 1, 2, 3 and 4 respectively. If $\theta = 20$, the corresponding probabilities are 0.20, 0.30, 0.36 and 0.14 respectively. Based on the single observation, we would like to test $H_0 : \theta = 10$ vs. $H_1 : \theta = 20$. Find a most powerful test of level 0.35 for this problem.

[7+7=14]

7. An experimenter wants to compare the effects of 2 levels of temperature and also the effects of 2 levels of pressure on a chemical process. In each run of the experiment, one level of temperature and one level of pressure can be used. Suggest a suitable design for this experiment in each of the following cases:

(a) four runs of the experiment are possible in one day and the experiment will run over two days,

(b) two runs of the experiment are possible in one day and the experiment will run over four days.

(It is known that all runs of the experiment in a single day can be carried out under similar conditions, but conditions may vary on different days).
In each case, state the null and alternative hypotheses of interest. Also give the form of the ANOVA table for analysing the data showing the sources of variation, associated degrees of freedom and expressions for the sum of squares. [14]

8. Consider a population with 3 units, labeled 1, 2 and 3. Let the values of a random variable of interest (y) for these units be y_1, y_2 and y_3 respectively. A simple random sample without replacement (SRSWOR) of size 2 is drawn from this population. Consider the two estimators:

(a) $\bar{y}$, i.e., the usual sample mean, and

(b) $\hat{\hat{Y}} = \begin{cases} \frac{y_1}{2} + \frac{y_2}{2} & \text{if the sample consists of units 1 and 2} \\ \frac{y_1}{2} + \frac{2y_3}{3} & \text{if the sample consists of units 1 and 3} \\ \frac{y_2}{2} + \frac{y_3}{3} & \text{if the sample consists of units 2 and 3.} \end{cases}$

(i) Show that both estimators are unbiased for the population mean.

(ii) Show that Var$(\hat{\hat{Y}}) <$ Var$(\bar{y})$ if $y_3(3y_2 - 3y_1 - y_3) > 0$.
(Hint: Suppose $\bar{x}$ is the sample mean for a simple random sample without replacement of size n from a population of size N with population unit values $x_1, \ldots, x_N$. Then

$$\text{Var}(\bar{x}) = \frac{N-n}{Nn} \frac{\sum_{i=1}^{N}(x_i - \bar{X})^2}{N-1},$$

where $\bar{X} = \frac{\sum_{i=1}^{N} x_i}{N}$.)

[7+7=14]

Section (iii): Physics

1. (a) Consider a hollow sphere of radius r having a surface density of mass equal to μ. Consider any point P inside the sphere which is at a distance a from the origin. Find the gravitational force and potential at P due to the mass of the sphere.

   (b) Let a particle P move under the central force of attraction directed towards a fixed point C. The trajectory of the particle is a circle passing through the fixed point C. Show that the force $\vec{F}$ on P is given by the law

$$\vec{F} \propto \frac{\vec{r}}{r^6},$$

   where $\vec{r}$ is the position vector $\overrightarrow{CP}$.

[5+9=14]

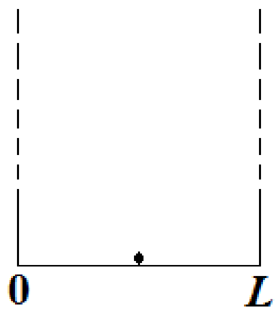
2. The wave function of a particle of mass m moving freely inside a one-dimensional infinite potential well of length L is given by

$$\psi(x) = \begin{cases} A\cos(\frac{\pi x}{L})\sin(\frac{2\pi x}{L}), & \text{inside the well} \\ 0, & \text{everywhere else.} \end{cases}$$

   (a) Calculate the value of A.

   (b) What is the probability that the particle is found in the left half of the box (i.e., in the region $0 \le x \le \frac{L}{2}$) on position measurement?

   (c) Calculate the uncertainty in momentum for the system.



[3+5+6=14]

3. A train passes a platform with velocity v. Two clocks are placed on the edge of the platform separated by a distance L and synchronized relative to an observer on the platform. The train runs in the direction from clock 1 to clock 2. Clock 1 reads time t_1 when it coincides with the front of the train and clock 2 reads time t_2 when it coincides with the rear of the train. Answer the following questions relative to *an observer on the train*.

   (a) What is the length of the train?

   (b) What is the reading of clock 2 when the clock 1 coincides with the front of the train?

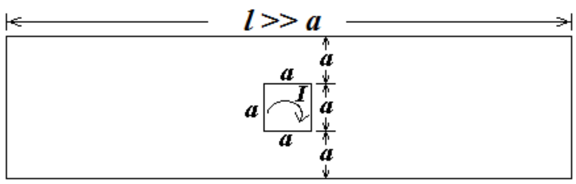
   (c) What is the time interval between the above two events, i.e., front of the train coinciding with clock 1 and rear of the train coinciding with clock 2?

[2+6+6=14]

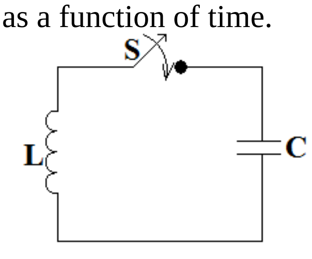
4. (a) A square loop of wire of side a, lies midway between two long wires, $3a$ apart, and in the same plane. Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected. A clockwise current I in the square loop is gradually increasing with $\frac{dI}{dt} = k$ where k is a constant, $k > 0$.

   (i) Find the emf induced in the bigger loop.

   (ii) Show the direction of flow of the induced current.



   (b) A capacitor is charged up to a potential V and connected to the inductor L. At time $t = 0$, the switch is closed. Find the current in the circuit as a function of time.



[8+6=14]

5. (a) The ground state of chlorine is $2_{P_{\frac{3}{2}}}$. Find the magnetic moment. Into how many sub-states will the ground state split in a weak magnetic field?

(b) The quantum number of two electrons in a two-valence electron are:

$$n_1 = 6,\ l_1 = 3,\ s_1 = \frac{1}{2}$$
$$n_2 = 5,\ l_2 = 1,\ s_2 = \frac{1}{2}$$

(i) Assuming L-S coupling, find the possible values of L, and hence J.

(ii) Assuming J-J coupling, find the possible values of J.

[7+7=14]

6. (a) A gas obeys the equation of state

$$P = \frac{T}{V} + \frac{B(T)}{V^2}$$

where $B(T)$ is a function that depends on temperature T only, and the other symbols have their usual meaning. The gas is initially at temperature T having a volume V_0. It is then expanded isothermally and reversibly to a volume $V_1 = 2V_0$. Find the work done doe to the expansion.

(b) The kinetic energy of a free rigid body in spherical polar coordinates is,

$$E_{kinetic} = \frac{1}{2}I(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2)$$

where I denotes the moment of inertia of the rigid body. Find the partition function Z of the system.
(Hint: You need to express the energy in terms of momenta and coordinate variables and then calculate Z.)

[7+7=14]

7. A series R-L-C circuit is excited from a constant-peak variable frequency voltage source of the form $V = V_0\ Sin\ \omega t$, where V_0 is constant. The current in the circuit becomes maximum at a frequency of $\omega_0 = 600$ rad/sec, and falls to half of the maximum value at ω = 400 rad/sec. If the resistance in the circuit is 3Ω, find L and C. [14]

8. (a) For a radix-6 number system, we need to represent each radix-6 digit in binary form. Design a binary code to represent all radix-6 digits such that the 5's complement for each digit is obtained just by complementing each bit in the binary code of that digit.

   (b) A Boolean function f^D generated from f by interchanging the operations '+' and '.' and the identity elements '0' and '1', is called the 'dual' of 'f'. A Boolean function 'f' is self-dual, if $f = f^D$.

   Check if the function $f(a, b, c) = b'(a'c' + ac) + b(a'c + ac')$ is self-dual or not.

[7+7=14]

Section (iv): Computer Science

1. (a) Let P be a program for computing the median of a set of n real numbers that runs in $O(n)$ time. Use P to design a divide and conquer algorithm for sorting a set of n real numbers that runs in $O(n \log n)$ time.

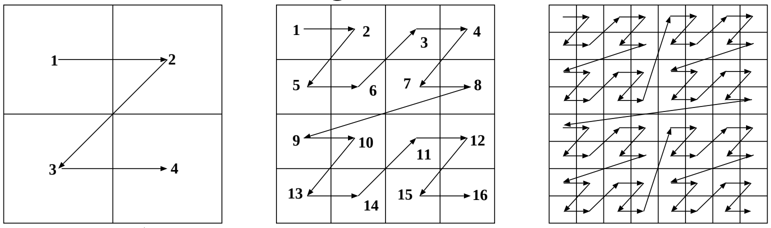
   (b) A sequence $S = \langle s_1, s_2, \ldots, s_n \rangle$ of n distinct real numbers is said to be a *unimodal* sequence if there exists an index i ($1 \leq i \leq n$) such that $s_1 < s_2 < \ldots < s_i > s_{i+1} > \ldots > s_n$. The element s_i is said to be the *mode* of the sequence S. Write an algorithm that can find the mode of a given sequence S in $O(\log n)$ time.

[6+8=14]

2. Let T be a binary tree having n nodes. Define $\Delta(x, y)$ to be the length of the path between two nodes $x, y \in T$. Design an efficient algorithm (preferably $O(n)$ time) for identifying two nodes $a, b \in T$ such that $\Delta(a, b) = max_{x,y \in T} \Delta(x, y)$, i.e., the path from a to b is the longest path in T.

   Note that a and b may both lie either in the left sub-tree of T, or in the right sub-tree of T, or one may lie in the left sub-tree while the other lies in the right sub-tree of T. [14]

3. Let A be an $N \times N$ matrix of integers, where $N = 2^m$ for some integer $m > 0$. Consider a program that traverses the matrix following a Z-like traversal, and prints the elements of the matrix in the order of traversal, as shown in the figures below.



$m = 1$ $m = 2$ $m = 3$

For $m = 2$, the output will be:

$$1, 2, 5, 6, 3, 4, 7, 8, 9, 10, 13, 14, 11, 12, 15, 16.$$

Note that when m increases by 1, a bigger Z is replaced by four copies of a smaller Z, and the smaller Zs are connected as shown above. This suggests that it is possible to use a recursive procedure to do the traversal.

The outline of the program is given below. Write the procedure `zed` in C, so that the program prints the elements of A in the required order.

```
void zed(int matrix[N][N], int start_row, int start_col,
         int end_row, int end_col)
{
  /* Fill in this missing part */
}

void main(void)
{
  /* declarations, initialization, etc. */
  zed(A, 0, 0, N-1, N-1);
}
```

[14]

4. Consider a jungle, where tigers and elephants live. The only source of water in the forest is a lake, situated in the middle of the forest. Tigers and elephants drink water from the lake. Suppose we model each animal by a process. The code outlines for tiger processes and elephant processes are given below. Using semaphores to synchronize their operation, fill in the outlines so that a tiger process and an elephant process never execute `drink_from_lake()` simultaneously.

Your solution should, however, permit multiple animals of the same kind to drink simultaneously. You may use as many semaphores as necessary.

```
shared semaphore ...;
```

```
tiger() {
...
drink_from_lake();
...
}
```

```
elephant() {
...
drink_from_lake();
...
}
```

[14]

5. (a) Consider the transactions T_1 and T_2 where T_2 is nested within T_1. If the underlying DBMS follows an immediate update policy for a log-maintenance system, suggest what actions would be taken by the recovery manager when a crash occurs in each of the four different positions shown below.

	T_1:	Read(A)
		Write(A)
Position 1 $\longrightarrow$		
	T_2:	Read(B)
		Write(B)
Position 2 $\longrightarrow$		
		Commit(T_2)
Position 3 $\longrightarrow$		
		Commit(T_1)
Position 4 $\longrightarrow$		

(b) For a relation $R(A, B, C, D, E, F)$, the following functional dependencies are given:

$\{AB \rightarrow C,\ \ BC \rightarrow D,\ \ CD \rightarrow E,\ \ A \rightarrow F\}$.

Decompose R into a set of normalized relations so that the new set of relations is free from partial and transitive dependencies.

[8+6=14]

6. (a) Let $\Sigma = \{0, 1\}$ and let

$D = \{w \mid w \in \Sigma^*$ and w has the same number of occurrences of the substrings 01 and 10 $\}$.

Thus, $101 \in D$ (it contains one occurrence each of 10 and 01), but $1010 \notin D$ (it contains two occurrences of 10 and one occurrence of 01). Show that D is a regular language.

(b) A grammar G is said to be *ambiguous* if for some string in the language generated by G, there exists two or more different parse trees. Show that the following grammar G_1 is ambiguous:

$$G_1 : \quad S \to D \; ; \; D \qquad D \to T \; L$$
$$T \to \textbf{int} \qquad T \to \textbf{char}$$
$$L \to L \; , \; L \qquad L \to I$$
$$I \to \textbf{id} \qquad I \to *\textbf{id}$$

S, D, T, L and I are non-terminals; all other symbols (**int**, **char**, **id**, '$*$', ';' and ',') are terminals.

Write down an **unambiguous** grammar G_2 such that G_2 generates the same language as G_1.

[8+6=14]

7. (a) Calculate the network address of a machine whose IP address is 172.18.139.25 and whose subnet mask is 255.255.192.0.

(b) Calculate the maximum throughput of the link layer service provided by an Ethernet operating at 10 Mbps, assuming an inter frame gap of 12 bytes. Also calculate the channel utilization. The sizes of the various frame parts are as follows:

MAC preamble	=	7 bytes
Start of frame delimiter	=	1 byte
Destination address	=	6 bytes
Source address	=	6 bytes
Length of frame	=	2 bytes
Check sequence	=	4 bytes
Payload	=	1212 bytes (maximum)

[5+5+4=14]

8. (a) Assume that there is a binary multiplier for multiplying two n-bit unsigned binary integers. The multiplication is performed in $O(n \log n)$ clock cycles. Let A and B be two floating point binary numbers each having an n-bit mantissa ($B \neq 0$). Devise an algorithm for computing $\frac{A}{B}$ by using repeated multiplication operations. You may assume that A and B are normalized, and that a subtraction operation requires one clock cycle. Also, compute the time complexity of your division algorithm in terms of clock cycles.

HINT: Write the mantissa of B as $1 - y$ where $0 < y < 1$.

(b) Consider a unifunction pipeline consisting of five stages, S_1, S_2, S_3, S_4 and S_5. The required function is evaluated in nine clock cycles following the reservation table as given below.

Clock cycle $\rightarrow$	0	1	2	3	4	5	6	7	8
S_1	X								X
S_2		X	X					X	
S_3				X					
S_4					X	X			
S_5							X	X	

Find the minimum average latency cycle of the above pipeline so that no collision occurs in accessing any stage of the pipeline during any clock cycle.

[7+7=14]

Section (v): Engineering and Technology

1. (a) The pressure developed in an Otto cycle having a compression ratio of 4 is 33 bar when the temperature is maximum during the cycle. If the temperature and pressure at the end of the suction cycle are 47 degree Celsius and 2 bar respectively, and the compression/expansion cycle follows the law $PV^2 =$ constant, calculate the maximum temperature of the Otto cycle.

   (b) Calculate the difference in temperature of water at the top and bottom of a waterfall 420 metre high assuming all energy to be converted into heat and retained by the water. Given J=4.2×10^7 ergs/cal, g=10 m/sec^2, specific heat of water is 1 cal/gm/degree Celsius.

[10+4=14]

2. (a) A wheel making five revolutions per second is uniformly slowed down to rest in four seconds. What angle does the wheel turn through during these four seconds?

(b) A particle starts from rest and moves along a circular path of radius R. The kinetic energy of the particle is given as $K = A.s$, where A is a constant and s is the distance travelled, measured along the circular arc. Show that the magnitude of the force acting on the particle when it has traversed a distance s, is given by

$$F = A\sqrt{1 + \frac{4s^2}{R^2}}.$$

[7+7=14]

3. Consider a hypothetical three-input gate $H(A, B, C) = AC' + B'C$. Realize B', AB and $A + B$ by using the minimum number of H gates. [3+5+6=14]

4. A magic square of order n is an arrangement of n^2 distinct integers from 1 to n^2 in an $n \times n$ square matrix such that the sum of n numbers in each row, each column and each of the two main diagonals equals to the same value, i.e., $\frac{n^3+n}{2}$. The magic square is said to be of odd order if n is odd. Magic squares of odd order can be generated as follows:

   Place the number 1 in the middle column of the last row. Subsequently, after placing a number m, the next number $m + 1$ is placed in the square one position below and to the left of m. If m is placed in the bottommost row, then $m+1$ is placed in the topmost row, and if m is placed in the leftmost column, then $m+1$ is placed in the rightmost column. If the position of $m + 1$ thus obtained is already filled, then $m+1$ is placed just above m (this position is guaranteed to be empty). An example with $n = 5$ is shown below.

9	2	25	18	11
3	21	19	12	10
22	20	13	6	4
16	14	7	5	23
15	8	1	24	17

   Write a C program to implement the above logic for an $n \times n$ square where n is odd. [14]

5. (a) Let A and B be two integers represented by 4 bits in "Sign-Magnitude" form. We compare A and B according to the following rule:

- A dominates B if $|A| > |B|$
- A equals B if $|A| = |B|$
- B dominates A if $|A| < |B|$

Design a combinational circuit to implement the above mentioned dominance relation.

(b) A Boolean operation denoted by '$\Rightarrow$', is defined as follows:

A	B	A$\Rightarrow$ B
T	T	T
F	T	T
T	F	F
F	F	T

Show that $A \vee B$ can be expressed in terms of $\Rightarrow$ alone, without using any 0 or 1 input.

[7+7=14]

6. Design a finite state sequential machine (FSM), if possible, for the following cases, assuming that the input is always a string of 0's and 1's of unknown length. In each case, if it is possible to design an FSM, then realize it with the minimum number of gates and flip-flops; otherwise explain why it is not possible.

(a) The FSM produces an output 1 if the input string has an equal number of 0's and 1's, and 0 otherwise.

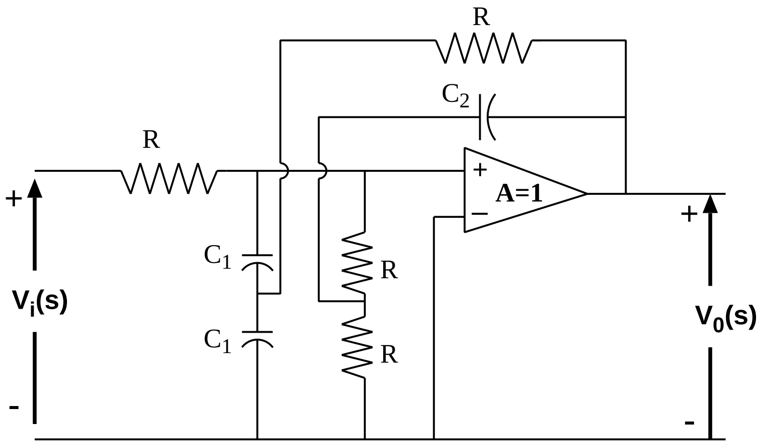
(b) The FSM produces an output 1 if the number of 0's in the input string is strictly greater than the number of 1's in it, and 0 otherwise.

(c) The FSM produces an output 1 if the (number of 0's mod 3) = (number of 1's mod 3) in the input string, and 0 otherwise.

[4+5+5=14]

7. A 300 KVA transformer has a core loss of 1.5 KW and a full load I^2R loss that is four times the core loss. Calculate the fraction of the full load at which the efficiency is maximized. If the value of this maximum efficiency is 0.9, then what is the power factor at which the machine is operating? [14]

8. Show that the following circuit acts as a bandpass filter with peak at a frequency $\omega_0 = \sqrt{\frac{2}{C_1\ C_2\ R^2}}$. Find the Q-factor of the filter.



[10+4=14]

9. Two horizontal perfectly conducting parallel plates A and B of infinite extent are spaced 1 cm apart. A stream of electrons is injected at a velocity corresponding to an accelerating voltage of 10 kV into the space between them through a hole in plate A at an angle of 30 degree to it. Calculate

   (a) the value and polarity of the potential difference which is required between A and B so that the electrons just touch the plate B.

   (b) the horizontal distance, measured from the point of injection, at which the electrons touch the plate B.

   (Assume $\frac{e}{m} = 18 \times 10^{10}$ in MKS units) [9+5=14]

10. A 2μF capacitor is discharged through a coil having an inductance of $2H$ and a resistance of 100Ω. The capacitor was initially charged to a voltage of 10V. Starting from the circuit equation

    (a) derive an expression for the current i through the coil,

    (b) check if the circuit is oscillatory,

    (c) compute the value of the additional resistance, if any, so as to make the above circuit critically damped.

    [8+3+3=14]

11. A single stage FET amplifier with a load resistance of $10\text{K}\Omega$ has a midband gain of 40 dB and a bandwidth of 20 kHz. The same gain is to be obtained by two identical cascaded stages, each having a load resistance of R. Calculate the value of R and the bandwidth of this 2-stage amplifier. In all cases, assume that the lower 3dB frequency is zero and the FETs used have the same characteristics. [7+7=14]

12. A 200 V, 13.77 KW shunt-wound d.c. motor is tested by the Swinburne method. The test produces the following readings:

    No-load running test:

    (a) Field current = 2 A
    (b) Armature current = 10 A

    Armature locked test:

    (a) Potential difference applied across the brushes = 5 V
    (b) Armature current = 50 A

    Calculate the full-load efficiency of the above d.c. motor. [14]