2016

Booklet No. TEST CODE: QMB

Afternoon

Questions: 8 **Time: 2 hours**

- On the answer booklet write your Name, Registration number, Test Centre, Test Code and the Number of this Booklet in the appropriate places on the Answer-sheet.
- The test QMB is of short answer type. It has altogether eight questions. A candidate has to answer all questions.

1. (a) If a, b, c are the sides of ΔABC and A, B, C are respectively the angles opposite to them, then find the value of

$$\begin{vmatrix} a^2 & b\sin A & c\cos A \\ b\sin A & 1 & \cos(B-C) \\ c\sin A & \cos(B-C) & 1 \end{vmatrix}$$

[10]

(b) For $x > 0$, if $[x]$ denotes the greatest integer less than or equal to x, then find the value of

$$\lim_{n\to\infty} \frac{[1^2x] + [2^2x] + [3^2x] + \cdots + [n^2x]}{n^3}.$$

[5]

2. (a) Find the values of a and b such that

$$\lim_{x\to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1.$$

[10]

(b) If $|x| < 1$ and $|y| < 1$ and $x \neq y$, then find the sum to infinity of the series

$$(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \cdots$$

[5]

3. (a) Let $\triangle_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$, where n is a positive integer, then show that $\sum_{a=1}^{n} \triangle_a = C$, a constant. [10]

(b) $2n$ boys are randomly divided into two groups each containing n boys. Find the probability that two particular boys X and Y will be in different groups. [5]

4. (a) Find the total number of ways in which a person can be given at least one rupee from four 25 paisa coins, three 50 paisa coins and two 1 rupee coins. [9]

(b) Show that the map $f : Q \to Q$ defined by $f(x) = 3x+2$ is one-one and onto, where Q is the set of rational numbers. Also find the formula for f^{-1}. [6]

5. (a) Find the remainder when 2^{75} is divided by 37. [5]

   (b) Let p, q, r be given real numbers. Solve the following system of equations

$$\begin{aligned} x + py + p^2 z &= p^3 \\ x + qy + q^2 z &= q^3 \\ x + ry + r^2 z &= r^3 \end{aligned}$$

   for x, y, z. [10]

6. (a) A balanced six faced die is rolled once. If the number of dots which show up is n, then a fair coin is tossed n times. Find the probability of seeing exactly 4 heads showing up in this experiment. [7]

   (b) Let $x_n, n = 1, 2, \ldots$ be a sequence of real numbers converging to a real number a as $n \to \infty$. Define

$$y_n = \begin{cases} x_n - \frac{1}{n} & \text{if } n \text{ is an odd integer;} \\ 2x_n & \text{if } n \text{ is an even integer.} \end{cases}$$

   Does y_n converge? What are its limit points? [8]

7. (a) Suppose a, b, c are in arithmetic progression and a^2, b^2, c^2, are in the geometric progression. If $a < b < c$ and $a + b + c = \frac{3}{2}$, then find the value of a. [5]

   (b) Let

$$p = \sum_{1 \le i < j \le n} \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j} \right) \text{ and } q = \sum_{1 \le i < j \le n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j} \right).$$

   Show that $q = \frac{n}{2} p$. [10]

8. (a) Let a system of homogeneous equations

$$\begin{aligned} tx + (t+1)y + (t-1)z &= 0 \\ (t+1)x + ty + (t+2)z &= 0 \\ (t-1)x + (t+2)y + tz &= 0 \end{aligned}$$

   For which value of t, the system has non-trivial solutions? [7]

   (b) Suppose $z > y > x > 0$ and $S = \frac{x^2+y^2+z^2}{x+y+z}$. Show that

$$\frac{x^2}{z} < S < \frac{z^2}{x}.$$

   [8]