2015

Booklet No. TEST CODE: QMB

Afternoon

Questions: 8 **Time: 2 hours**

- On the answer booklet write your Name, Registration number, Test Centre, Test Code and the Number of this Booklet in the appropriate places on the Answer-sheet.
- The test QMB is of short answer type. It has altogether ten questions. A candidate has to answer any eight questions.

1. (a) Evaluate $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{3^i 3^j 3^k}$ for $(i \neq j \neq k)$.

   (b) A restaurant offers 5 choices of appetizer, 10 choices of main meal and 4 choices of dessert. A customer can choose to eat just one course, or two different courses, or all three courses. Assuming all choices are available, how many different possible meals does the restaurant offer? [10+5 = 15]

2. (a) Given a function g which has derivative $g'(x)$ for all x satisfying $g'(0) = 2$ and $g(x+y) = e^y g(x) + e^x g(y)$ for all $x, y \ \varepsilon \ R$.
   Show that $g'(x) + g(x) - 2e^x = 0$.

   (b) Let $A = \begin{bmatrix} \cos(\theta+\alpha) & \sin(\theta+\alpha) & 1 \\ \cos(\theta+\beta) & \sin(\theta+\beta) & 1 \\ \cos(\theta+\gamma) & \sin(\theta+\gamma) & 1 \end{bmatrix}$.
   Show that $|A|$ is independent of θ. [10+5 = 15]

3. (a) Show that $1 = \int\int\int x^{l-1} y^{m-1} z^{n-1} (1-x-y-z)^{p-1} dx \ dy \ dz$; $(l, m, n, p) \geq 1$ taken over the tetrahedron bounded by the planes, $x = 0, y = 0, z = 0, x + y + z = 1$ is $\frac{(\Gamma l)(\Gamma m)(\Gamma n)(\Gamma p)}{\Gamma(l+m+n+p)}$.

   (b) Consider the following two experiments. In the first experiment a fair coin is tossed six times. Let p be the probability of getting exactly four heads in a row in this experiment. In the second experiment a pair of fair coins are simultaneously tossed three times. Let q be the probability of getting a pair of heads exactly twice in a row in the second experiment. then p and q are related as in

      i. $p = q$
      ii. $p < q$
      iii. $p > q$
      iv. $p = 2q$ [8+7 = 15]

4. (a) A rectangle is inscribed in an acute angled triangle ABC with one side of the rectangle along the base BC and one vertex in AB and another vertex in AC. Find the dimensions of the rectangle having the maximum area.

   (b) For all real numbers x, y show that

      i. $|x+y| \leq |x| + |y|$

      ii. $|x-y| \geq |x| - |y|$ [10+5 = 15]

5. (a) If $f(x) = (x) - x$, where $(x) =$ the least integer greater than or equal to x, then show that $f(x)$ is continuous at all non-integral values of x.

   (b) If one root of the equation $x^2 + ax + b = 0$ is also a root of $x^2 + mx + n = 0$, show that its other root is a root of $x^2(2a - m)x + a^2 - am + n = 0$. [10+5 = 15]

6. (a) Solve the equation $2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$.

   (b) A spherical iron ball of radius 10cm coated with a layer of ice of uniform thickness melts at a rate of $100\pi cm^3/min$. Find the rate at which the thickness of the ice decreases when the thickness of ice is 5cm. [10+5 = 15]

7. (a) For $n = 1, 2, 3, ...$ let A_n be a sequence of regular convex polygons with 2^{n+1} sides and radius (distance from any vertex to the center) 1. Thus A_1 is a square with diagonal length 2 and A_2 is an octagon with diagonal length 2. Let a_n be the area of the polygon A_n. Then

      i. Express a_n in terms of the angle θ_n made by the lines joining any two nearest vertices to the center of A_n,

      ii. Find the limit of the sequence a_n as $n \to \infty$.

   (b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, Let B be a 3×3 matrix $[B_1\ B_2\ B_3]$ such that $A^{50}B_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}$, $A^{50}B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $A^{50}B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
   Find the value of $|B|$. [10+5 = 15]

8. (a) Use the principle of mathematical induction to prove that,
   ${}^nC_0 +^n C_1 +^n C_2 + ... +^n C_n = 2^n$ for all $n \varepsilon N$.

   (b) Examine the following function for continuity at the origin
   $f(x) = \left\{ \begin{array}{cc} \frac{xe^{1/x}}{1+e^{1/x}} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{array} \right\}$. [10+5 = 15]

9. (a) If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then show that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$.

   (b) Evaluate $\int_{-2}^{3} |1 - x^2| dx$. [10+5 = 15]

10. (a) Find the value of $\Sigma_{n=0}^{\infty} \frac{2n+3}{3^n}$.

   (b) If $m > 2$ and $t \in R$, find the integral part of $(\frac{4|t|}{16+t^2})^m$. [8+7 = 15]