PEA 2015 (Mathematics)

Answer all questions

1. Consider the polynomial $P(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \{1, 2, \ldots, 9\}$. If $P(10) = 5861$, then the value of c is

   (a) 1.

   (b) 2.

   (c) 6.

   (d) 5.

2. Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ be a twice continuously differentiable function, and $x^* \in A$ be such that $\frac{\partial f}{\partial x}(x^*) = 0$.

   (a) $\frac{\partial^2 f}{\partial x^2}(x^*) \leq 0$ is a *sufficient* condition for x^* to be a point of local maximum of f on A;

   (b) $\frac{\partial^2 f}{\partial x^2}(x^*) \leq 0$ is a *necessary* condition for x^* to be a point of local maximum of f on A;

   (c) $\frac{\partial^2 f}{\partial x^2}(x^*) \leq 0$ is *necessary and sufficient* for x^* to be a point of local maximum of f on A;

   (d) $\frac{\partial^2 f}{\partial x^2}(x^*) \leq 0$ is *neither necessary nor sufficient* for x^* to be a point of local maximum of f on A.

3. You are given five observations x_1, x_2, x_3, x_4, x_5 on a variable x, ordered from lowest to highest. Suppose x_5 is increased. Then,

   (a) The mean, median, and variance, all increase.

   (b) The median and the variance increase but the mean is unchanged.

   (c) The variance increases but the mean and the median are unchanged.

(d) None of the above.

4. Suppose the sum of coefficients in the expansion $(x+y)^n$ is 4096. The largest coefficient in the expansion is:

   (a) 924.

   (b) 1024.

   (c) 824.

   (d) 724.

5. There are three cards. The first is green on both sides, the second is red on both sides and the third is green on one side and red on the other. I choose a card with equal probability, then a side of that card with equal probability. If the side I choose of the card is green, what is the probability that the other side is green?

   (a) $\frac{1}{3}$.

   (b) $\frac{1}{2}$.

   (c) $\frac{2}{3}$.

   (d) $\frac{3}{4}$.

6. The value of

$$\int_0^{\frac{\pi}{2}} x \sin x dx$$

   is:

   (a) 0.

   (b) -1.

   (c) $\frac{1}{2}$.

   (d) 1

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} ax + b & \text{if } x \geq 0 \\ \sin 2x & \text{if } x < 0 \end{cases}$$

For what values of a and b is f continuous but *not* differentiable?

(a) $a = 2$, $b = 0$.

(b) $a = 2$, $b = 1$.

(c) $a = 1$, $b = 1$.

(d) $a = 1$, $b = 0$.

8. A student wished to regress household food consumption on household income. By mistake the student regressed household income on household food consumption and found R^2 to be 0.35. The R^2 in the correct regression of household food consumption on household income is

(a) 0.65.

(b) 0.35.

c) $1 - (.35)^2$.

(d) None of the above.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

.

Which of the following statements is true?

(a) $(x = 1, y = 0)$ is a local maximum of f.

(b) $(x = 1, y = 0)$ is a local minimum of f.

(c) $(x = 1, y = 0)$ is neither a local maximum nor a local minimum of f.

(d) $(x = 0, y = 0)$ is a global maximum of f.

10. Let

$$f(x) = \frac{x + \sqrt{3}}{1 - \sqrt{3}x}$$

for all $x \neq \frac{1}{\sqrt{3}}$. What is the value of $f(f(x))$?

(a) $\frac{x-\sqrt{3}}{1+\sqrt{3}x}$.

(b) $\frac{x^2+2\sqrt{3}x+3}{1-2\sqrt{3}x+3x}$.

(c) $\frac{x+\sqrt{3}}{1-\sqrt{3}x}$.

(d) $\frac{x+\sqrt{3}}{1-\sqrt{3}x}$.

11. The continuous random variable X has probability density $f(x)$ where

$$f(x) = \begin{cases} a & \text{if } 0 \leq x < k \\ b & \text{if } k \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Then $E(X)$ is given by:

(a) $\frac{b(1-a)^2}{2a(a-b)}$.

(b) $\frac{1}{2}$.

(c) $\frac{a-b}{(a+b)}$.

(d) $\frac{1-2b+ab}{2(a-b)}$.

12. The set of values of x for which $x^2 - 3|x| + 2 < 0$ is given by:

(a) $\{x : x < -2\} \cup \{x : x > 1\}$.

(b) $\{x : -2 < x < -1\} \cup \{x : 1 < x < 2\}$.

(c) $\{x : x < -1\} \cup \{x : x > 2\}$.

(d) None of the above.

13. The system of linear equations

$$\begin{aligned} (4d-1)x + y + z &= 0 \\ -y + z &= 0 \\ (4d-1)z &= 0 \end{aligned}$$

has a non-zero solution if:

(a) $d = \frac{1}{4}$.

(b) $d = 0$.

(c) $d \neq \frac{1}{4}$.

(d) $d = 1$.

14. Suppose F is a cumulative distribution function of a random variable x distributed in $[0, 1]$ defined as follows:

$$F(x) = \begin{cases} ax + b & \text{if } x \geq a \\ x^2 - x + 1 & \text{otherwise} \end{cases}$$

where $a \in (0, 1)$ and b is a real number. Which of the following is true?

(a) F is continuous in $(0, 1)$.

(b) F is differentiable in $(0, 1)$.

(c) F is not continuous at $x = a$.

(d) None of the above.

15. The solution of the optimization problem

$$\begin{aligned}
&\max_{x,y}\ 3xy - y^3 \\
&\text{subject to} \\
&2x + 5y \geq 20 \\
&x - 2y = 5 \\
&x, y \geq 0.
\end{aligned}$$

is given by:

(a) $x = 19, y = 7$.

(b) $x = 45, y = 20$.

(c) $x = 15,\ y = 5$.

(d) None of the above.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Let g be the inverse of the function f. If $f'(1) = g(1) = 1$, then $g'(1)$ equals to

(a) 0.

(b) $\frac{1}{2}$.

(c) -1.

(d) 1.

17. Consider a quadratic polynomial $P(x)$. Suppose $P(1) = -3, P(-1) = -9, P(-2) = 0$. Then, which of the following is true.

(a) $P(\frac{1}{2}) = 0$.

(b) $P(\frac{5}{2}) = 0$.

(c) $P(\frac{5}{4}) = 0$.

(d) $P(\frac{3}{4}) = 0$.

18. For any positive integers k, ℓ with $k \geq \ell$, let $C(k, \ell)$ denote the number of ways in which ℓ distinct objects can be chosen from k objects. Consider $n \geq 3$ distinct points on a circle and join every pair of points by a line segment. If we pick three of these line segments uniformly at random, what is the probability that we choose a triangle?

(a) $\frac{C(n,2)}{C(C(n,2),3)}$.

(b) $\frac{C(n,3)}{C(C(n,2),3)}$

(c) $\frac{2}{n-1}$.

(d) $\frac{C(n,3)}{C(C(n,2),2)}$.

19. Let $X = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1, 2x + \frac{y}{2} \leq 1, x \geq 0, y \geq 0\}$. Consider the optimization problem of maximizing a function $f(x) = ax + by$, where a, b are real numbers, subject to the constraint that $(x, y) \in X$. Which of the following is not an optimal value of f for any value of a and b?

(a) $x = 0, y = 1$.

(b) $x = \frac{1}{3}, y = \frac{2}{3}$.

(c) $x = \frac{1}{4}, y = \frac{1}{4}$.

(d) $x = \frac{1}{2}, y = 0$.

20. Let $F : [0, 1] \to \mathbb{R}$ be a differentiable function such that its derivative $F'(x)$ is increasing in x. Which of the following is true for every $x, y \in [0, 1]$ with $x > y$?

(a) $F(x) - F(y) = (x - y)F'(x)$.

(b) $F(x) - F(y) \geq (x - y)F'(x)$.

(c) $F(x) - F(y) \leq (x - y)F'(x)$.

(d) $F(x) - F(y) = F'(x) - F'(y)$.

21. A bag contains N balls of which a ($a < N$) are red. Two balls are drawn from the bag without replacement. Let p_1 denote the probability that the first ball is red and p_2 the probability that the second ball is red. Which of the following statements is true?

(a) $p_1 > p_2$.

(b) $p_1 < p_2$.

(c) $p_2 = \frac{a-1}{N-1}$.

(d) $p_2 = \frac{a}{N}$.

22. Let $t = x + \sqrt{x^2 + 2bx + c}$ where $b^2 > c$. Which of the following statements is true?

(a) $\frac{dx}{dt} = \frac{t-x}{t+b}$.

(b) $\frac{dx}{dt} = \frac{t+2x}{2t+b}$.

(c) $\frac{dx}{dt} = \frac{1}{2x+b}$.

(d) None of the above.

23. Let A be an $n \times n$ matrix whose entry on the i-th row and j-th column is $\min(i, j)$. The determinant of A is:

(a) n.

(b) 1.

(c) $n!$

(d) 0.

24. What is the number of non-negative integer solutions of the equation $x_1+x_2+x_3 = 10$?

(a) 66.

(b) 55.

(c) 100.

(d) None of the above.

25. The value of

$$\int_b^{2b} \frac{xdx}{x^2+b^2},$$

$b > 0$ is:

(a) $\frac{1}{b}$.

(b) $\ln 4b^2$.

(c) $\frac{1}{2}\ln(\frac{5}{2})$.

(d) None of the above.

26. Let f and g be functions on $\mathbb{R}^2$ defined respectively by

$$f(x,y) = \tfrac{1}{3}x^3 - \tfrac{3}{2}y^2 + 2x,$$

and

$$g(x,y) = x - y.$$

Consider the problems of *maximizing* and *minimizing* f on the constraint set $C = \{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}$.

(a) f has a maximum at $(x = 1, y = 1)$, and a minimum at $(x = 2, y = 2)$.

(b) f has a maximum at $(x = 1, y = 1)$, but does not have a minimum.

(c) f has a minimum at $(x = 2, y = 2)$, but does not have a maximum.

(d) f has neither a maximum nor a minimum.

27. A particular men's competition has an unlimited number of rounds. In each round, every participant has to complete a task. The probability of a participant completing the task in a round is p. If a participant fails to complete the task in a round, he is eliminated from the competition. He participates in every round before being eliminated. The competition begins with three participants. The probability that all three participants are eliminated in the same round is:

(a) $\frac{(1-p)^3}{1-p^3}$.

(b) $\frac{1}{3}(1-p)$.

(c) $\frac{1}{p^3}$.

(d) None of the above.

28. Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely. The probability that each husband sits next to his wife is:

(a) $\frac{2}{15}$.

(b) $\frac{1}{3}$.

(c) $\frac{4}{15}$.

(d) None of the above.

29. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function. For every $x, y, z \in \mathbf{R}$, we know that $f(x,y) + f(y,z) + f(z,x) = 0$. Then, for every $x, y \in \mathbf{R}^2$, $f(x,y) - f(x,0) + f(y,0) =$

(a) 0.

(b) 1.

(c) -1.

(d) None of the above.

30. The minimum value of the expression below for $x > 0$ is:

$$\frac{\left(x+\frac{1}{x}\right)^6 - \left(x^6+\frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}$$

(a) 1.

(b) 3.

(c) 6

(d) 12.