2014

BOOKLET No. TEST CODE : PMB

Afternoon

Duration of test : 2 hours

Write your Registration number, Test Code, Number of this booklet, etc. in the appropriate places on your ANSWER BOOKLET.

This test has questions arranged in two groups.

Each group consists of 6 questions.

You need to answer 4 questions FROM EACH GROUP.

Each question carries 10 marks. Total marks = 80.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET
AND/OR ON YOUR ANSWER BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP ! WAIT FOR THE SIGNAL TO START

INSTRUCTIONS FOR CANDIDATES

- Please answer FOUR questions from EACH group.
- Each question carries 10 marks. Total marks : 80.
- $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ denote respectively the set of real numbers, set of complex numbers, set of all integers and set of all positive integers.

Group A

1. Let f be a twice differentiable function on $(0, 1)$. It is given that for all $x \in (0, 1)$, $|f''(x)| \le M$ where M is a non-negative real number. Prove that f is uniformly continuous on $(0, 1)$.

2. Let f be a real-valued continuous function on $[0, 1]$ which is twice continuously differentiable on $(0, 1)$. Suppose that $f(0) = f(1) = 0$ and f satisfies the following equation:
$$x^2 f''(x) + x^4 f'(x) - f(x) = 0.$$
   (a) If f attains its maximum M at some point x_0 in the open interval $(0, 1)$, then prove that $M = 0$.
   (b) Prove that f is identically zero on $[0, 1]$.

3. Consider the set S consisting of all Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{N}$ for all n. Is the set S countable? Justify your answer.

4. Let A be a compact subset of $\mathbb{R} \setminus \{0\}$ and B be a closed subset of $\mathbb{R}^n$. Prove that the set $\{a \cdot b \mid a \in A, b \in B\}$ is closed in $\mathbb{R}^n$.

5. Does there exist a continuous function $f : [0, 1] \to [0, \infty)$ such that $\displaystyle\int_0^1 x^n f(x)\, dx = 1$ for all $n \ge 1$? Justify your answer.

6. Prove that there exists a constant $c > 0$ such that for all $x \in [1, \infty)$,
$$\sum_{n\ge x} \frac{1}{n^2} \le \frac{c}{x}.$$

Group B

1. Let $(\mathbb{Q}, +)$ be the group of rational numbers under addition. If G_1, G_2 are nonzero subgroups of $(\mathbb{Q}, +)$, then prove that $G_1 \cap G_2 \neq \{0\}$.

2. With proper justifications, examine whether there exists any surjective group homomorphism

   (a) from the group $(\mathbb{Q}(\sqrt{2}), +)$ to the group $(\mathbb{Q}, +)$,

   (b) from the group $(\mathbb{R}, +)$ to the group $(\mathbb{Z}, +)$.

3. Consider the ring

$$R = \left\{ \frac{2^k m}{n} \;\middle|\; m, n \text{ odd integers; } k \text{ is a non-negative integer} \right\}.$$

   (a) Describe all the units (invertible elements) of R.

   (b) Demonstrate one nonzero proper ideal I of R.

   (c) Examine whether the ideal I that you have chosen, is a prime ideal of R (that is, whether $a \cdot b \in I$ implies $a \in I$ or $b \in I$).

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $T^2 = 0$. If r denotes the rank of T (that is, $r = \dim(\text{Image}(T))$), then show that $r \leq \frac{n}{2}$.

5. Let A be a 2×2 matrix with real entries such that $\text{Tr}(A) = 0$ and $\det(A) = -1$.

   (a) Prove that there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

   (b) Suppose that T is a 2×2 real matrix with respect to the above basis such that $TA = AT$. Prove that T is a diagonal matrix with respect to that basis.

6. Let $i = \sqrt{-1}$ and $\alpha = i + \sqrt{2}$. Construct a polynomial $f(x)$ with integer coefficients such that $f(\alpha) = 0$.